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A cluster expansion approach to the Heilmann-Lieb liquid crystal model

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Abstract A monomer-dimer model with a short-range attractive interaction favoring colinear dimers is considered on the lattice $\mathbb{Z}^2$. Although our choice of the chemical potentials results in more horizontal than vertical dimers, the horizontal dimers have no long-range translational order - in agreement with the Heilmann-Lieb conjecture [10].

Keywords Monomer-Dimer · Liquid Crystal · Heilmann-Lieb Conjecture · Cluster Expansion

Introduction

A liquid crystal, at low temperatures, displays a long-range order in the orientation of its molecules, while there is no complete ordering in their positions. In this paper we present a model characterized by these two features. In particular we consider a monomer-dimer model on the two-dimensional lattice $\mathbb{Z}^2$ characterized by different chemical potentials for horizontal and vertical dimers ($\mu_{\mathrm{h}} > \mu_{\mathrm{v}}$ to fix ideas) and by a short-range potential $J > 0$ that favors collinear dimers. We prove that when the parameters satisfy

$$\mu_{\mathrm{h}} > -J \quad \text{and} \quad \mu_{\mathrm{v}} < -\frac{5}{2}\, J \,, \tag{0.1}$$

the system has the properties of a liquid crystal.

Onsager [14] was the first to propose hard-rods models in order to explain the existence of liquid crystals. In 1970 Heilmann and Lieb [8,9] studied systems of monomer and dimers (hard-rods of length 2) interacting only via the hard-core potential, and proved the absence of phase transitions in great generality. Then in 1972 they [10] proposed two monomer-dimer models (named

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I and II) on the lattice $\mathbb{Z}^2$, where short-range attractive interactions among parallel dimers are considered beyond the hard-core interaction. Heilmann and Lieb claimed that these systems are liquid crystals. In particular they proved the presence of a phase transition, by means of a reflection positivity argument: at low temperature there is orientational order. Moreover they conjectured the absence of complete translational ordering for their models. A proof of this conjecture for the model I was announced in [10] by Heilmann and Kjær, but never appeared. Letawe, in her thesis [12], claimed to prove the conjecture by cluster expansion methods, even if the result has never been published in a journal. Letawe's polymers are built starting from contours and the major difficulty seems to arise when she has to deal with a polymer lying in the interior of another one: the two polymers would not be independent. To overcome this problem, ratios of partition functions with different (horizontal or vertical) boundary conditions $Z^{\mathrm{v}}/Z^{\mathrm{h}}$ are introduced, but it is not proved that these ratios are sufficiently small to guarantee the convergence of the cluster expansion.

Numerical simulations related to the Heilmann-Lieb conjecture are performed in [15]. We also mention that, in absence of attractive interaction, systems of sufficiently long hard-rods were proved to display a phase transition and behave like liquid crystals by Disertori and Giuliani [3], using a two scales cluster expansion and the Pirogov-Sinai theory.

In the present paper we study a model obtained from the model I of Heilmann and Lieb [10], but while they suppose

$$\mu_{\mathrm{h}} = \mu_{\mathrm{v}} =: \mu \quad \text{and} \quad \mu > -J \,, \tag{0.2}$$

we assume very different horizontal and vertical potentials as in (0.1). This choice of the parameters allows us to work with cluster expansion methods, by defining our polymers starting from regions of vertical dimers, instead of contours. The cluster expansion method permits to rewrite the logarithm of the partition function of a polymer system as a power series of the polymer activities. This expansion entails analyticity results and simplifies considerably the study of the correlation functions, which can be expressed in terms ratios of partition functions. Clearly the cluster expansion cannot hold in general on the whole space of parameters: it converges only when the polymer activities are small enough to compete with the entropy. A rigorous study of the conditions of convergence dates back to [6, 7, 16], by means of Kirkwood-Salsburg type of equations. In this paper we use a criterion proposed by Kotecky and Preiss [11] in 1986. Afterwards this criterion was compared to the previous ones, was improved and simplified in [1, 2, 4, 5, 13, 17] (for a clear and modern treatment we suggest for example the last work).

The paper is organized as follows. In the section 1 we introduce the model and we state the main results about its liquid crystal properties. In the section 2 we show how to rewrite the partition function as a suitable polymer partition function, following in part the ideas of [12]: our polymers turn out to be connected families of *regions* of vertical dimers and *lines* of horizontal dimers and monomers. In the section 3 we prove that the Kotecky-Preiss condition for the convergence of the cluster expansion is verified when the parameters satisfy (0.1) and the temperature is sufficiently low. Finally in

the section 4 we use the previous sections to prove the results stated in the section 1. The appendix A contains the study of a 1-dimensional monomer-dimer model, that is needed in the section 2. For the sake of completeness, in the appendix B we state the general results of cluster expansion needed in the paper.

1 Definitions and Main Results: the Model and its Liquid Crystal Properties

A *monomer-dimer configuration* on $\mathbb{Z}^2$ can be represented by a bonds[1] occupation vector $\alpha \in \{0,1\}^{B(\mathbb{Z}^2)}$ with hard-core interaction, namely:

$$\sum_{y \sim x} \alpha_{(x,y)} \;\leq\; 1 \quad \forall x \in \mathbb{Z}^2\,. \tag{1.1}$$

If $\alpha_{(x,y)} = 1$, we say that there is a *dimer* on the bond (x,y), or also that there is a dimer at the site x; if instead $\alpha_{(x,y)} = 0$ for all $y \sim x$, we say that there is a *monomer* on the site x. Dimers on $\mathbb{Z}^2$ may have two different orientations: vertical (*v-dimers*) or horizontal (*h-dimers*), according to the orientation of the occupied bond[2]. The model studied in the present paper favors one orientation of the dimers (the horizontal one), both via a chemical potential and via a short-range imitation.

Let Λ be a finite sub-lattice of $\mathbb{Z}^2$. Consider a *horizontal boundary condition*[3], namely we assume that every site of $\mathbb{Z}^2 \setminus \Lambda$ has a h-dimers (with either free or fixed positions). Denote by $\mathscr{D}^{\mathrm{h}}_{\Lambda}$ the set of monomer-dimer configurations on Λ (we allow also dimers toward the exterior[4]) which are compatible with the selected horizontal boundary condition.

The Hamiltonian, or energy, of a monomer-dimer configuration is defined as

$$\begin{aligned} H_\Lambda := {} & \frac{\mu_{\mathrm{h}} + J}{2}\,\#\left\{\begin{matrix}\text{sites of } \Lambda \text{ with}\\ \text{monomer}\end{matrix}\right\} + \frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2}\,\#\left\{\begin{matrix}\text{sites of } \Lambda \text{ with}\\ \text{v-dimer}\end{matrix}\right\} + \\ & + \frac{J}{2}\left(\#\left\{\begin{matrix}\text{sites of } \bar\Lambda \text{ with h-dimer}\\ \text{but h-neighbor also to a}\\ \text{v-dimer or a monomer}\end{matrix}\right\} + \#\left\{\begin{matrix}\text{sites of } \bar\Lambda \text{ with v-dimer}\\ \text{but v-neighbor also to a}\\ \text{h-dimer or a monomer}\end{matrix}\right\}\right). \end{aligned} \tag{1.2}$$

We assume that the parameters appearing in the Hamiltonian satisfy

$$\mu_{\mathrm{h}} > -J\,, \quad \mu_{\mathrm{h}} \geq \mu_{\mathrm{v}}\,, \quad J > 0\,. \tag{1.3}$$

[1] Two sites $x = (x_{\mathrm{h}}, x_{\mathrm{v}})$, $y = (y_{\mathrm{h}}, y_{\mathrm{v}}) \in \mathbb{Z}^2$ are *neighbors* ($x \sim y$) if $|x_{\mathrm{h}} - y_{\mathrm{h}}| + |x_{\mathrm{v}} - y_{\mathrm{v}}| = 1$. A pair of sites (x,y) is a *bond* if x,y are neighbors. $B(\mathbb{Z}^2)$ denotes the set of bonds.

[2] Two sites $x = (x_{\mathrm{h}}, x_{\mathrm{v}})$, $y = (y_{\mathrm{h}}, y_{\mathrm{v}}) \in \mathbb{Z}^2$ are *h-neighbors* if $x_{\mathrm{v}} = y_{\mathrm{v}}$ and $|x_{\mathrm{h}} - y_{\mathrm{h}}| = 1$, they are *v-neighbors* if $x_{\mathrm{h}} = y_{\mathrm{h}}$ and $|x_{\mathrm{v}} - y_{\mathrm{v}}| = 1$. A bond $(x,y) \in B(\mathbb{Z}^2)$ is *horizontal* if x,y are h-neighbors, it is *vertical* if x,y are v-neighbors.

[3] The *external boundary* of Λ is $\partial^{\mathrm{ext}}\Lambda := \{x \in \mathbb{Z}^2 \setminus \Lambda \,|\, x \text{ neighbor of } y \in \Lambda\}$. The *internal boundary* of Λ is instead $\partial\Lambda \equiv \partial^{\mathrm{int}}\Lambda := \{x \in \Lambda \,|\, x \text{ neighbor of } y \in \mathbb{Z}^2 \setminus \Lambda\}$. We set $\bar\Lambda := \Lambda \cup \partial^{\mathrm{ext}}\Lambda$.

[4] Namely we allow dimers having one endpoint in Λ and one in $\mathbb{Z}^2 \setminus \Lambda$.

In this way, if the horizontal boundary condition with free positions is chosen[5], then the *ground states* in $\mathscr{D}^{\mathrm{h}}_{\Lambda}$ (i.e. the configurations minimizing the energy under the given condition) are exactly the configurations where every site has a h-dimer. The partition function of the system is

$$Z^{\mathrm{h}}_{\Lambda} := \sum_{\alpha \in \mathscr{D}^{\mathrm{h}}_{\Lambda}} e^{-\beta H_{\Lambda}(\alpha)} \tag{1.4}$$

where the parameter $\beta > 0$ is the inverse temperature.

Remark 1.1 We want to show that the Hamiltonian (1.2) essentially corresponds to the model I introduced by Heilmann and Lieb in [10], except for the important fact that we allow the horizontal and vertical dimer potentials μ_{h}, μ_{v} to be different, while they take $\mu_{\mathrm{h}} = \mu_{\mathrm{v}} = \mu$. We can introduce another Hamiltonian (that maybe is written in a more natural way; see fig.1):

$$\begin{aligned} \widetilde{H}_{\Lambda} := &- \mu_{\mathrm{h}} \,\#\{\text{h-dimers in } \Lambda\} - \mu_{\mathrm{v}} \,\#\{\text{v-dimers in } \Lambda\} + \\ &- J \,\#\left\{\begin{matrix}\text{pairs of neighboring} \\ \text{colinear dimers in } \Lambda\end{matrix}\right\} \end{aligned} \tag{1.5}$$

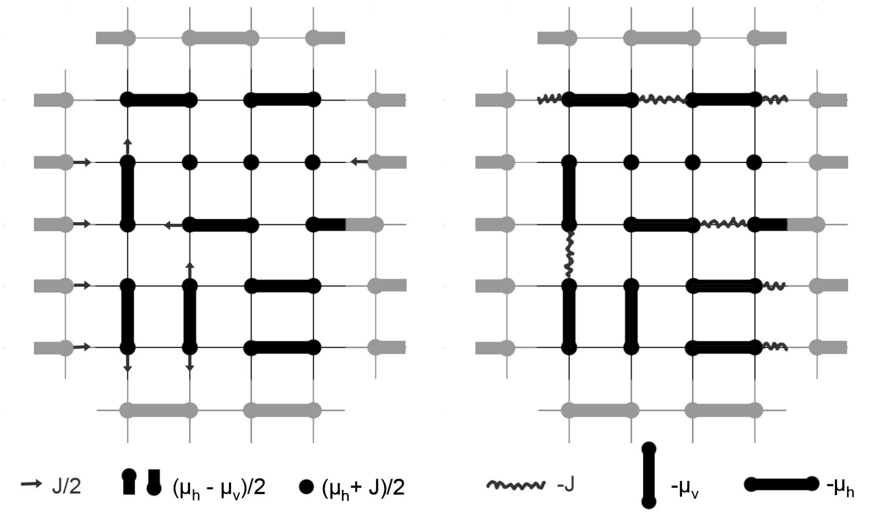

Fig. 1: The same monomer-dimer configuration on the lattice Λ and the corresponding energies in accordance to the Hamiltonian (1.2) (on the left) *and to the Hamiltonian* (1.5) (on the right). *A horizontal boundary condition is drawn in grey.*

The monomer-dimer model I in [10] is given by the Hamiltonian (1.5) with $\mu_{\mathrm{h}} = \mu_{\mathrm{v}} = \mu$, when Λ is a rectangular lattice of even sides lengths with

[5] Also fixed positions work, provided that the positions of the two h-dimers at the endpoints of each horizontal line of Λ allow a pure dimer configuration on that line.

periodic boundary conditions (torus). It is easy to show that when Λ is a torus the two Hamiltonians (1.2), (1.5) describe the same model; indeed they only differ by an additive constant which does not affect the Gibbs measure:

$$\widetilde{H}_\Lambda + \frac{\mu_\mathrm{h} + J}{2}\,|\Lambda| \;=\; H_\Lambda \tag{1.6}$$

since

$$\begin{aligned} |\Lambda| - 2\,\#\{\text{h-dimers in } \Lambda\} &= |\Lambda| - \#\left\{\begin{smallmatrix}\text{sites in } \Lambda \text{ with}\\ \text{h-dimer}\end{smallmatrix}\right\} = \\ &= \#\left\{\begin{matrix}\text{sites in } \Lambda \text{ with}\\ \text{monomer}\end{matrix}\right\} + \#\left\{\begin{matrix}\text{sites in } \Lambda \text{ with}\\ \text{v-dimer}\end{matrix}\right\}\;; \end{aligned}$$

$$2\,\#\{\text{v-dimers in } \Lambda\} \;=\; \#\left\{\begin{matrix}\text{sites in } \Lambda \text{ with}\\ \text{v-dimer}\end{matrix}\right\}\;;$$

$$|\Lambda| - 2\,\#\left\{\begin{matrix}\text{pairs of neighboring}\\ \text{colinear dimers in } \Lambda\end{matrix}\right\} = |\Lambda| - \#\left\{\begin{matrix}\text{sites in } \Lambda \text{ with h-dimer (v-dimer)}\\ \text{and h-neighbor (v-neighbor) to}\\ \text{another h-dimer (v-dimer)}\end{matrix}\right\} =$$

$$= \#\left\{\begin{matrix}\text{sites in } \Lambda \text{ with}\\ \text{monomer}\end{matrix}\right\} + \#\left\{\begin{matrix}\text{sites in } \Lambda \text{ with h-dimer (v-dimer)}\\ \text{and h-neighbor (v-neighbor) also}\\ \text{to something different}\end{matrix}\right\}\;.$$

On the other hand when Λ has horizontal boundary conditions the two Hamiltonians (1.2), (1.5) are not exactly equivalent. Indeed it holds[6]

$$\widetilde{H}_\Lambda + \frac{\mu_\mathrm{h} + J}{2}\,|\Lambda| + \frac{J}{2}\,\#\left\{\begin{matrix}\text{sites} \quad \text{in} \quad \partial_\mathrm{v}^{\mathrm{int}}\Lambda\\ \text{without h-dimer}\end{matrix}\right\} \;=\; H_\Lambda \tag{1.7}$$

when the following conventions are adopted in the definition (1.5): if only half a dimer is in Λ while the other half is in $\mathbb{Z}^2 \setminus \Lambda$, it counts $\frac{1}{2}$; if only one dimer of a pair of neighboring colinear dimers is in Λ, while the other one is in $\mathbb{Z}^2 \setminus \Lambda$, this pair counts $\frac{1}{2}$.

The monomer-dimer model that we have introduced, in a certain region of the parameters corresponding to large horizontal potential, small vertical potential and low temperature, behaves like a *liquid crystal*. This means that the model exhibits an order in the orientation of the molecules (dimers), while there is no complete order in their positions.

The following results will give a precise mathematical meaning to these statements. First we introduce some observables attached to the sites, asking questions as “Is there a horizontal dimer at site x?”, “If so, is it positioned to the left or to the right of x?”. To measure the absence or presence of some kind of order, at a microscopic level we study the expectations and the covariances of these quantities according to the Gibbs measure, while at a macroscopic level we introduce a suitable *order parameter* and study its expectation and possibly its variance[7].

[6] ∂_v, ∂_h denote respectively the *vertical*, *horizontal component* of the boundary; e.g. $\partial_\mathrm{v}\Lambda := \{x \in \Lambda \,|\, x \text{ h-neighbor of } y \in \mathbb{Z}^2 \setminus \Lambda\}$ and $\partial_\mathrm{h}\Lambda := \{x \in \Lambda \,|\, x \text{ v-neighbor of } y \in \mathbb{Z}^2 \setminus \Lambda\}$.

[7] When the expectation of the order parameter is zero but the variance is not, a small perturbation can lead to a spontaneous order of the system.

Define the following local observables[8]

$$f_{\mathrm{h},x} := \mathbb{1}\big(x \text{ has a h-dimer}\big) \ , \quad f_{\mathrm{v},x} := \mathbb{1}\big(x \text{ has a v-dimer}\big) \ ; \tag{1.8}$$

$$f_{\mathrm{l},x} := \mathbb{1}\big(x \text{ has a left-dimer}\big) \ , \quad f_{\mathrm{r},x} := \mathbb{1}\big(x \text{ has a right-dimer}\big) \ . \tag{1.9}$$

Clearly $f_{\mathrm{h},x} = f_{\mathrm{l},x} + f_{\mathrm{r},x}$ and $f_{\mathrm{h},x} + f_{\mathrm{v},x} \leq 1$. In the following we denote the Gibbs expectation of any observable f by

$$\langle f \rangle^{\mathrm{h}}_{\Lambda} := \frac{1}{Z^{\mathrm{h}}_{\Lambda}} \sum_{\alpha \in \mathscr{D}^{\mathrm{h}}_{\Lambda}} f(\alpha)\, e^{-\beta H_{\Lambda}(\alpha)} \ .$$

We denote by N the minimal distance between any two vertical components of the boundary of Λ and our only assumption on the shape of Λ is that $N \to \infty$ as $\Lambda \nearrow \mathbb{Z}^2$. To fix ideas one could think that Λ is a rectangle (in this case N would be simply its horizontal side length), but actually we will need to consider also non-simply connected regions.

There exists $\beta_0 > 0$ depending on μ_{h}, μ_{v}, J only and $N_0(\beta)$ depending on β, μ_{h}, J only such that the following results hold true.

Theorem 1.2 (Microscopic expectations) *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite having $N > N_0(\beta)$. Let $x \in \Lambda$ such that $\mathrm{dist}_{\mathrm{h}}(x, \partial\Lambda) > N_0(\beta)$. Then*

$$\langle f_{\mathrm{l},x} \rangle^{\mathrm{h}}_{\Lambda} \geq \frac{1}{2} - e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \quad , \quad \langle f_{\mathrm{r},x} \rangle^{\mathrm{h}}_{\Lambda} \geq \frac{1}{2} - e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \ . \tag{1.10}$$

As a consequence:

$$\langle f_{\mathrm{h},x} \rangle^{\mathrm{h}}_{\Lambda} \geq 1 - 2\, e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \ ; \tag{1.11}$$

$$\big| \langle f_{\mathrm{r},x} \rangle^{\mathrm{h}}_{\Lambda} - \langle f_{\mathrm{l},x} \rangle^{\mathrm{h}}_{\Lambda} \big| \leq 2\, e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \ . \tag{1.12}$$

Theorem 1.3 (Microscopic covariances) *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite such that $N > N_0(\beta)$. Let $x, y \in \Lambda$ such that $\mathrm{dist}_{\mathrm{h}}(x, \partial\Lambda) > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(y, \partial\Lambda) > N_0(\beta)$ and $\mathrm{dist}_{\mathrm{h}}(x, y) > N_0(\beta)$. Then:*

$$\big| \langle f_{\mathrm{l},x}\, f_{\mathrm{l},y} \rangle^{\mathrm{h}}_{\Lambda} - \langle f_{\mathrm{l},x} \rangle^{\mathrm{h}}_{\Lambda} \langle f_{\mathrm{l},y} \rangle^{\mathrm{h}}_{\Lambda} \big| \leq \frac{9m}{16}\, e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \ , \tag{1.13}$$

$$\big| \langle f_{\mathrm{r},x}\, f_{\mathrm{r},y} \rangle^{\mathrm{h}}_{\Lambda} - \langle f_{\mathrm{r},x} \rangle^{\mathrm{h}}_{\Lambda} \langle f_{\mathrm{r},y} \rangle^{\mathrm{h}}_{\Lambda} \big| \leq \frac{9m}{16}\, e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \ , \tag{1.14}$$

$$\big| \langle f_{\mathrm{l},x}\, f_{\mathrm{r},y} \rangle^{\mathrm{h}}_{\Lambda} - \langle f_{\mathrm{l},x} \rangle^{\mathrm{h}}_{\Lambda} \langle f_{\mathrm{r},y} \rangle^{\mathrm{h}}_{\Lambda} \big| \leq \frac{9m}{16}\, e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \ . \tag{1.15}$$

The definition of m is clarified in the Appendix (lemma A.5); anyway it can be sufficient to know that $m = e^{-\beta \frac{\mu_{\mathrm{h}}+3J}{2}} \big(1 + o(1)\big)$ as $\beta \to \infty$.

[8] We say that the site x has a *left-dimer* if there is a dimer on the bond $\big(x, x-(1,0)\big)$, a *right-dimer* if there is a dimer on the bond $\big(x, x+(1,0)\big)$.

The density of lattice sites occupied by h-dimers/v-dimers is respectively:

$$\nu_{\mathrm{h}} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f_{\mathrm{h},x} \quad , \quad \nu_{\mathrm{v}} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f_{\mathrm{v},x} \ . \tag{1.16}$$

A parameter measuring the orientational order of the dimers is

$$\Delta_{\text{orient.}} := \nu_{\mathrm{h}} - \nu_{\mathrm{v}} \ . \tag{1.17}$$

Corollary 1.4 (Orientational Order Parameter) *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite, having $N > 2\, N_0(\beta)$. Then*

$$\langle \Delta_{\text{orient.}} \rangle_{\Lambda}^{\mathrm{h}} \geq \left(1 - 2\, \frac{N_0(\beta)}{N}\right) \left(1 - 4\, e^{-\beta \frac{\mu_{\mathrm{h}} + J}{2}}\right) . \tag{1.18}$$

Hence

$$\lim_{\beta \nearrow \infty} \liminf_{\Lambda \nearrow \mathbb{Z}^2} \langle \Delta_{\text{orient.}} \rangle_{\Lambda}^{\mathrm{h}} = 1 \ . \tag{1.19}$$

The corollary 1.4 shows that fixing β sufficiently large and then choosing Λ sufficiently big (more precisely the distance N between vertical components of $\partial\Lambda$ must be large enough), the average density of sites occupied by h-dimers is arbitrarily close to 1 : in other terms the system is oriented along the horizontal direction.

The majority of sites is occupied by h-dimers. But there can still be some freedom, indeed we may distinguish the h-dimers in two classes according to their positions: a *h-dimer* is called *even* (resp. *odd*) if its left endpoint has even (resp. odd) horizontal coordinate. The density of lattice sites occupied by even/odd h-dimers is respectively:

$$\begin{aligned} \nu_{\text{even}} &:= \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathbb{1}\big(x \text{ has an even h-dimer}\big) = \frac{2}{|\Lambda|} \sum_{\substack{x \in \Lambda \\ x_{\mathrm{h}} \text{ even}}} f_{\mathrm{r},x} \ , \\ \nu_{\text{odd}} &:= \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathbb{1}\big(x \text{ has an odd h-dimer}\big) = \frac{2}{|\Lambda|} \sum_{\substack{x \in \Lambda \\ x_{\mathrm{h}} \text{ even}}} f_{\mathrm{l},x} \ . \end{aligned} \tag{1.20}$$

A parameter measuring the translational order of the h-dimers is

$$\Delta_{\text{transl.}} := \nu_{\text{even}} - \nu_{\text{odd}} \ . \tag{1.21}$$

Corollary 1.5 (Translational Order Parameter. Part I) *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite such that $N > 2\, N_0(\beta)$. Then*

$$\left| \langle \Delta_{\text{transl.}} \rangle_{\Lambda}^{\mathrm{h}} \right| \leq \left(1 - 2 \frac{N_0(\beta)}{N}\right) 2\, e^{-\beta \frac{\mu_{\mathrm{h}} + J}{2}} + 2\, \frac{N_0(\beta)}{N} \tag{1.22}$$

Hence

$$\lim_{\beta \nearrow \infty} \limsup_{\Lambda \nearrow \mathbb{Z}^2} \left| \langle \Delta_{\text{transl.}} \rangle_{\Lambda}^{\mathrm{h}} \right| = 0 \ . \tag{1.23}$$

Corollary 1.6 (Translational Order Parameter. Part II) *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite such that $N > 2\,N_0(\beta)$. Then*

$$\big\langle (\Delta_{\mathrm{transl.}})^2 \big\rangle_\Lambda^{\mathrm{h}} - \big(\langle \Delta_{\mathrm{transl.}} \rangle_\Lambda^{\mathrm{h}}\big)^2 \;\le\; \frac{1}{|\Lambda|}\,\frac{9m}{(1-e^{-\frac{m}{4}})^2} \;+\; \frac{N_0(\beta)}{N}\Big(6 - 8\frac{N_0(\beta)}{N}\Big)\,. \tag{1.24}$$

Hence for fixed $\beta > \beta_0$

$$\lim_{\Lambda \nearrow \mathbb{Z}^2} \big\langle (\Delta_{\mathrm{transl.}})^2 \big\rangle_\Lambda^{\mathrm{h}} - \big(\langle \Delta_{\mathrm{transl.}} \rangle_\Lambda^{\mathrm{h}}\big)^2 \;=\; 0\,. \tag{1.25}$$

The corollaries 1.5, 1.6 show that fixing β sufficiently large and then choosing Λ sufficiently big (in particular the distance between different components of $\partial_{\mathrm{v}}\Lambda$ must be big enough), the mean value and the variance of the difference between the density of even h-dimers and the density of odd h-dimers are arbitrarily close to zero. In other terms, at large but finite β, there is not a spontaneous translational order for the h-dimers.

Remark 1.7 The bounds (1.22) hold for any kind of horizontal boundary conditions, but in some particular cases it is possible to obtain a better result by a symmetry argument. Assume that Λ is a rectangle with $N+1$ sites in each horizontal side. If $N+1$ is odd, by choosing *horizontal dimers with free positions at the boundary* one obtains

$$\langle \Delta_{\mathrm{transl.}} \rangle_\Lambda^{\mathrm{h}} \;=\; \langle \nu_{\mathrm{even}} \rangle_\Lambda^{\mathrm{h}} - \langle \nu_{\mathrm{odd}} \rangle_\Lambda^{\mathrm{h}} \;=\; 0 \tag{1.26}$$

for all parameters β, J, μ_{h}, μ_{v}. To prove it consider the reflection on Λ with respect to the vertical axis at distance $\frac{N}{2}$ from $\partial_{\mathrm{v}}\Lambda$: this transformation induces a bijection $T : \mathscr{D}_\Lambda^{\mathrm{h}} \to \mathscr{D}_\Lambda^{\mathrm{h}}$. It is easy to check that $H_\Lambda(T(\alpha)) = H_\Lambda(\alpha)$, $\nu_{\mathrm{even}}(T(\alpha)) = \nu_{\mathrm{odd}}(\alpha)$, $\nu_{\mathrm{odd}}(T(\alpha)) = \nu_{\mathrm{even}}(\alpha)$ for all $\alpha \in \mathscr{D}_\Lambda^{\mathrm{h}}$.
On the other hand if $N+1$ is even, by choosing *periodic boundary conditions* one still obtains

$$\langle \Delta_{\mathrm{transl.}} \rangle_\Lambda^{\mathrm{per.}} \;=\; 0 \tag{1.27}$$

for all parameters β, J, μ_{h}, μ_{v}. To prove it one can consider the reflection on Λ with respect to two vertical axis at distance $\frac{N+1}{2}$ from each other: it induces a bijection from $\mathscr{D}_\Lambda^{\mathrm{per.}}$ to itself having all the previous properties.

2 Polymer Representation

In this section we show how to rewrite the partition function Z_Λ^{h} as a polymer partition function of type (B.1). This representation will be suitable for applying the cluster expansion machinery (see Appendix B) in a regime of large horizontal potential, small vertical potential and low temperature.

We start by isolating the "few" vertical dimers. Associate to each monomer-dimer configuration $\alpha \in \mathscr{D}_\Lambda^{\mathrm{h}}$ the set

$$V = V(\alpha) := \{x \in \Lambda \mid x \text{ has a v-dimer according to } \alpha\}\,.$$

Partition V into its connected components (as a sub-graph of the lattice[9] $\mathbb{Z}^2$):

$$V = \bigcup_{i=1}^{n} S_i \quad , \quad S_i \in \mathscr{S}_\Lambda \;\forall\, i \quad , \quad \mathrm{dist}_{\mathbb{Z}^2}(S_i, S_j) > 1 \;\forall\, i \neq j$$

where the family $\mathscr{S}_\Lambda$ is defined by

$$\begin{aligned} S \in \mathscr{S}_\Lambda \;\overset{\text{def}}{\Leftrightarrow}\; & S \subseteq \Lambda \,,\; S \neq \emptyset \,,\; S \text{ connected (as a sub-graph of } \mathbb{Z}^2) \,, \\ & \text{every maximal vertical segment of } S \text{ has an even number} \\ & \text{of sites} \,, \\ & S \text{ does not contains those sites of } \partial_{\mathrm{v}}^{\mathrm{int}}\Lambda \text{ that necessarily} \\ & \text{have a h-dimer because of the boundary conditions.} \end{aligned} \tag{2.1}$$

The knowledge of the set V (or equivalently of $S_1, \ldots, S_n$) does not determine completely the configuration α of the system, since on $\Lambda \setminus V$ there can be both h-dimers and monomers. Anyway a fundamental feature of the model is that the system on $\Lambda \setminus V$ can be partitioned into independent 1-dimensional systems. Introduce the family $\mathscr{L}_\Lambda(V)$ defined by

$$L \in \mathscr{L}_\Lambda(V) \;\overset{\text{def}}{\Leftrightarrow}\; L \text{ is a maximal horizontal line of } \Lambda \setminus V \,. \tag{2.2}$$

The Hamiltonian (1.2) rewrites as

$$\begin{aligned} H_\Lambda \;=\; & \sum_{i=1}^{n} \left(\frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2}\, |S_i| \;+\; \frac{J}{2}\, |\partial_{\mathrm{h}} S_i| \;+\; \frac{J}{2}\, |\partial_{\mathrm{v}} S_i \cap \partial\Lambda| \right) \;+ \\ & + \sum_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} \left(\frac{\mu_{\mathrm{h}} + J}{2}\, \#\left\{ \begin{matrix} \text{sites of } L \text{ with} \\ \text{monomer} \end{matrix} \right\} \;+\; \frac{J}{2}\, \#\left\{ \begin{matrix} \text{sites of } L \text{ with h-dimer} \\ \text{but h-neighbor also to a} \\ \text{monomer or to } \cup_i S_i \end{matrix} \right\} \right) . \end{aligned}$$

Hence the partition function (1.4) rewrites as (see fig.2)

$$Z_\Lambda^{\mathrm{h}} \;=\; \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{S_1, \ldots, S_n \in \mathscr{S}_\Lambda \\ \mathrm{dist}(S_i, S_j) > 1 \,\forall i \neq j}} \prod_{i=1}^{n} e^{-\beta \left(\frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} |S_i| \,+\, \frac{J}{2} |\partial_{\mathrm{h}} S_i| \,+\, \frac{J}{2} |\partial_{\mathrm{v}} S_i \cap \partial\Lambda| \right)} \prod_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} Z_L \tag{2.3}$$

where Z_L is the monomer-dimer partition function of the line L, considered as a sub-lattice of the 1-dimensional lattice $\mathbb{Z}$, with suitable boundary conditions:

$$Z_L \;:=\; \sum_{\alpha_L \in \mathscr{D}_L} e^{-\beta H_L(\alpha_L)} \; e^{I_{\mathrm{l}, x_{\mathrm{l}}}(\alpha_{x_{\mathrm{l}}})} \; e^{I_{\mathrm{r}, x_{\mathrm{r}}}(\alpha_{x_{\mathrm{r}}})} \,. \tag{2.4}$$

[9] On any graph the distance between two objects is defined as the length of the shortest path connecting them. In particular $\mathrm{dist}_{\mathbb{Z}^2}(S, S') := \inf_{x \in S,\, y \in S'} \mathrm{dist}_{\mathbb{Z}^2}(x, y)$ for all $S, S' \subset \mathbb{Z}^2$ and $\mathrm{dist}_{\mathbb{Z}^2}(x, y) := |x_{\mathrm{h}} - y_{\mathrm{h}}| + |x_{\mathrm{v}} - y_{\mathrm{v}}|$ for all $x = (x_{\mathrm{h}}, x_{\mathrm{v}})$, $y = (y_{\mathrm{h}}, y_{\mathrm{v}}) \in \mathbb{Z}^2$.

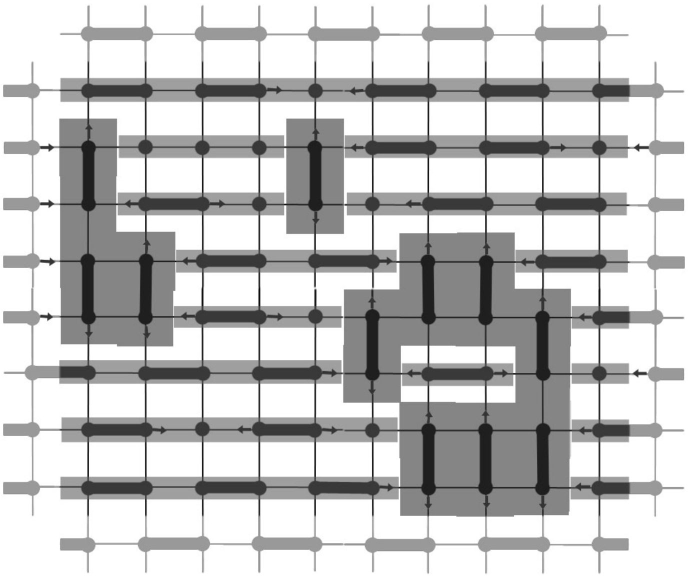

Fig. 2: A monomer-dimer configuration on Λ and the corresponding regions S_1, S_2, S_3 *and* lines $L_1, \dots, L_{15} \in \mathscr{L}_\Lambda(\cup_i S_i)$. *Given the positions of the regions, the configurations on the lines are mutually independent: the* arrows *represent the energy contributions of type $J/2$. A horizontal boundary condition is drawn.*

An explanation of the notations introduced in (2.4) is required. $\mathscr{D}_L$ denotes the set of monomer-dimer configurations on L (dimers can only be horizontal, external dimers at the endpoints of L are allowed);

$$H_L \;:=\; \frac{\mu_{\mathrm{h}} + J}{2}\;\#\left\{\begin{matrix}\text{sites of } L \text{ with}\\ \text{monomer}\end{matrix}\right\} \;+\; \frac{J}{2}\;\#\left\{\begin{matrix}\text{sites of } L \text{ with dimer}\\ \text{but h-neighbor also to}\\ \text{a monomer}\end{matrix}\right\}\;;$$

x_{l}, x_{r} denote respectively the left, right endpoint of the line L (which eventually may coincide): observe[10] that because of (2.2)

$$\bigcup_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} x_{\mathrm{l}}(L) \;=\; \Big(\big(\cup_i\, \partial^{\mathrm{ext}}_{\mathrm{r}} S_i\big) \cap \Lambda \Big) \;\sqcup\; \Big(\partial_{\mathrm{l}}\Lambda \setminus \cup_i\, \partial_{\mathrm{l}} S_i \Big)\;, \tag{2.5}$$

$$\bigcup_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} x_{\mathrm{r}}(L) \;=\; \Big(\big(\cup_i\, \partial^{\mathrm{ext}}_{\mathrm{l}} S_i\big) \cap \Lambda \Big) \;\sqcup\; \Big(\partial_{\mathrm{r}}\Lambda \setminus \cup_i\, \partial_{\mathrm{r}} S_i \Big)\;; \tag{2.6}$$

[10] ∂_{l}, ∂_{r} denote respectively the *left*, *right component* of the vertical boundary; e.g. $\partial_{\mathrm{l}}\Lambda := \{x \in \Lambda \,|\, x - (1,0) \in \mathbb{Z}^2 \setminus \Lambda\}$ and $\partial_{\mathrm{r}}\Lambda := \{x \in \Lambda \,|\, x + (1,0) \in \mathbb{Z}^2 \setminus \Lambda\}$.

finally[11]

$$
\begin{array}{l}
\text{if } x_{\mathrm{l}} \in \cup_i\, \partial^{\mathrm{ext}}_{\mathrm{r}} S_i \ \Rightarrow\ I_{\mathrm{l},x_{\mathrm{l}}} := \big(-\infty\ -\beta\tfrac{J}{2}\ 0\big) \\
\text{if } x_{\mathrm{l}} \in \partial_{\mathrm{l}}\Lambda\,,\ \text{on } x_{\mathrm{l}}-(1,0) \text{ it is fixed a l-dimer } \Rightarrow\ I_{\mathrm{l},x_{\mathrm{l}}} := \big(-\infty\ 0\ -\beta\tfrac{J}{2}\big) \\
\text{if } x_{\mathrm{l}} \in \partial_{\mathrm{l}}\Lambda\,,\ \text{on } x_{\mathrm{l}}-(1,0) \text{ it is fixed a r-dimer } \Rightarrow\ I_{\mathrm{l},x_{\mathrm{l}}} := \big(0\ -\infty\ -\infty\big) \\
\text{if } x_{\mathrm{l}} \in \partial_{\mathrm{l}}\Lambda\,,\ \text{on } x_{\mathrm{l}}-(1,0) \text{ there is a free h-dimer } \Rightarrow\ I_{\mathrm{l},x_{\mathrm{l}}} := \big(0\ 0\ -\beta\tfrac{J}{2}\big)
\end{array}
\tag{2.7}
$$

and, similarly,

$$
\begin{array}{l}
\text{if } x_{\mathrm{r}} \in \cup_i\, \partial^{\mathrm{ext}}_{\mathrm{l}} S_i \ \Rightarrow\ I_{\mathrm{r},x_{\mathrm{r}}} := \big(-\beta\tfrac{J}{2}\ -\infty\ 0\big) \\
\text{if } x_{\mathrm{r}} \in \partial_{\mathrm{r}}\Lambda\,,\ \text{on } x_{\mathrm{r}}+(1,0) \text{ it is fixed a r-dimer } \Rightarrow\ I_{\mathrm{r},x_{\mathrm{r}}} := \big(0\ -\infty\ -\beta\tfrac{J}{2}\big) \\
\text{if } x_{\mathrm{r}} \in \partial_{\mathrm{r}}\Lambda\,,\ \text{on } x_{\mathrm{r}}+(1,0) \text{ it is fixed a l-dimer } \Rightarrow\ I_{\mathrm{r},x_{\mathrm{r}}} := \big(-\infty\ 0\ -\infty\big) \\
\text{if } x_{\mathrm{r}} \in \partial_{\mathrm{r}}\Lambda\,,\ \text{on } x_{\mathrm{r}}+(1,0) \text{ there is a free h-dimer } \Rightarrow\ I_{\mathrm{r},x_{\mathrm{r}}} := \big(0\ 0\ -\beta\tfrac{J}{2}\big)\,.
\end{array}
\tag{2.8}
$$

The 1-dimensional systems described by Z_L, $L \in \mathscr{L}_\Lambda(\cup_i S_i)$, are studied in the Appendix A.

In the form (2.3) of Z^{h}_Λ, the weight of the regions $(S_1,\ldots,S_n)$ is not a product of the weights of each region S_i, because of the lines L connecting different regions. Therefore the regions $S_i \in \mathscr{S}_\Lambda$ are not a good choice for a polymer representation of the model. In order to decouple some regions from some other ones, it is possible to do a simple trick. It is convenient to deal in different ways with the endpoints lying on $\partial^{\mathrm{ext}} S_i$ and those on $\partial\Lambda$; hence given a line $L \in \mathscr{L}_\Lambda(\cup_i S_i)$ we set

$$
\begin{aligned}
\varepsilon_{\mathrm{l},x_{\mathrm{l}}} &:= \mathbb{1}\big(x_{\mathrm{l}} \in (\cup_i\, \partial^{\mathrm{ext}}_{\mathrm{r}} S_i) \cap \Lambda\big)\ , \quad \eta_{\mathrm{l},x_{\mathrm{l}}} := 1-\varepsilon_{\mathrm{l},x_{\mathrm{l}}} \overset{(2.5)}{=} \mathbb{1}\big(x_{\mathrm{l}} \in (\partial_{\mathrm{l}}\Lambda) \setminus \cup_i\, \partial_{\mathrm{l}} S_i\big)\ ; \\
\varepsilon_{\mathrm{r},x_{\mathrm{r}}} &:= \mathbb{1}\big(x_{\mathrm{r}} \in (\cup_i\, \partial^{\mathrm{ext}}_{\mathrm{l}} S_i) \cap \Lambda\big)\ , \quad \eta_{\mathrm{r},x_{\mathrm{r}}} := 1-\varepsilon_{\mathrm{r},x_{\mathrm{r}}} \overset{(2.6)}{=} \mathbb{1}\big(x_{\mathrm{r}} \in (\partial_{\mathrm{r}}\Lambda) \setminus \cup_i\, \partial_{\mathrm{r}} S_i\big)\ .
\end{aligned}
$$

Using the notations of the Appendix A, given a line $L \in \mathscr{L}_\Lambda(\cup_i S_i)$ we introduce the two vectors representing the boundary conditions outside its endpoints x_{l}, x_{r} :

$$
B_{\mathrm{l},x_{\mathrm{l}}} := \Big(e^{I_{\mathrm{l},x_{\mathrm{l}}}(l)}\ e^{I_{\mathrm{l},x_{\mathrm{l}}}(r)}\ e^{-\beta\frac{\mu_{\mathrm{h}}+J}{4} + I_{\mathrm{l},x_{\mathrm{l}}}(m)} \Big)\ ,\ B_{\mathrm{r},x_{\mathrm{r}}} := \begin{pmatrix} e^{I_{\mathrm{r},x_{\mathrm{r}}}(l)} \\ e^{I_{\mathrm{r},x_{\mathrm{r}}}(r)} \\ e^{-\beta\frac{\mu_{\mathrm{h}}+J}{4} + I_{\mathrm{r},x_{\mathrm{r}}}(m)} \end{pmatrix}\ ;
$$

then to shorten the notation we set

$$
b_{\mathrm{l},x_{\mathrm{l}}} := \frac{1}{\sqrt{\lambda_1}}\, B_{\mathrm{l},x_{\mathrm{l}}}\, E^{(1)}_{\mathrm{r}}\ , \quad b_{\mathrm{r},x_{\mathrm{r}}} := \frac{1}{\sqrt{\lambda_1}}\, E^{(1)}_{\mathrm{l}} B_{\mathrm{r},x_{\mathrm{r}}}\ .
$$

[11] The possible states of a site $x \in L$ are three: "l"=*left-dimer* namely a dimer on the bond $\big(x, x-(1,0)\big)$, "r"=*right-dimer* namely a dimer on the bond $\big(x, x+(1,0)\big)$, "m"=*monomer*. Here we think $I_{\mathrm{l},x_{\mathrm{l}}}$, $I_{\mathrm{r},x_{\mathrm{r}}}$ as vectors: $I_{\mathrm{l},x_{\mathrm{l}}} = (I_{\mathrm{l},x_{\mathrm{l}}}(l)\ I_{\mathrm{l},x_{\mathrm{l}}}(r)\ I_{\mathrm{l},x_{\mathrm{l}}}(m))$ and $I_{\mathrm{r},x_{\mathrm{r}}} = (I_{\mathrm{r},x_{\mathrm{r}}}(l)\ I_{\mathrm{r},x_{\mathrm{r}}}(r)\ I_{\mathrm{r},x_{\mathrm{r}}}(m))$.

Now define

$$R_L := \frac{Z_L}{\lambda_1^{|L|}\, b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}}} - b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}} \tag{2.9}$$

and, using $\mathscr{L}$ as an abbreviation for $\mathscr{L}_\Lambda(\cup_i S_i)$, rewrite the quantity $\prod_{L\in\mathscr{L}} Z_L$ by means of elementary algebraic tricks:

$$\begin{aligned}\prod_{L\in\mathscr{L}} \frac{Z_L}{\lambda_1^{|L|}} &= \prod_{L\in\mathscr{L}} \left(\left(R_L + b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}}\right) b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}}\right) \\ &= \left(\prod_{L\in\mathscr{L}} b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}}\right) \sum_{\mathscr{K}\subseteq\mathscr{L}} \left(\prod_{L\in\mathscr{K}} R_L\right) \left(\prod_{L\in\mathscr{L}\setminus\mathscr{K}} b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}}\right) .\end{aligned}$$

By identities (2.5), (2.6) it holds

$$\begin{aligned}\prod_{L\in\mathscr{L}} b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}} &= \left(\prod_{x\in\partial_\mathrm{l}\Lambda\setminus\cup_i\partial_\mathrm{l}S_i} b_{\mathrm{l},x}\right)\left(\prod_{x\in\partial_\mathrm{r}\Lambda\setminus\cup_i\partial_\mathrm{r}S_i} b_{\mathrm{r},x}\right) \\ \prod_{L\in\mathscr{L}\setminus\mathscr{K}} b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}} &= \left(\prod_{\substack{x\in(\cup_i\partial_\mathrm{r}^{\mathrm{ext}}S_i)\cap\Lambda\\ x\notin \operatorname{supp}\mathscr{K}}} b_{\mathrm{l},x}\right)\left(\prod_{\substack{x\in(\cup_i\partial_\mathrm{l}^{\mathrm{ext}}S_i)\cap\Lambda\\ x\notin \operatorname{supp}\mathscr{K}}} b_{\mathrm{r},x}\right) ;\end{aligned}$$

By substituting into the previous formula and thinking $\mathscr{K} = \{L_1, \dots, L_p\}$, we find out[12]

$$\begin{aligned}\prod_{L\in\mathscr{L}} \frac{Z_L}{\lambda_1^{|L|}} &= \left(\prod_{x\in\partial_\mathrm{v}\Lambda\setminus\cup_i\partial_\mathrm{v}S_i} b_{\mathrm{l/r},\,x}\right)\cdot \\ &\cdot\sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{L_1,\dots,L_p\in\mathscr{L}\\ L_h\neq L_k\ \forall h\neq k}} \left(\prod_{k=1}^{p} R_{L_k}\right)\left(\prod_{\substack{x\in(\cup_i\partial_\mathrm{v}^{\mathrm{ext}}S_i)\cap\Lambda\\ x\notin\cup_k L_k}} b_{\mathrm{r/l},\,x}\right) .\end{aligned} \tag{2.10}$$

[12] In the first product on the r.h.s. of (2.10) the shorten notation $b_{\mathrm{l/r},x}$ means: take $b_{\mathrm{l},x}$ if $x \in \partial_\mathrm{l}\Lambda$, take $b_{\mathrm{r},x}$ if $x \in \partial_\mathrm{r}\Lambda$; notice that $\partial_\mathrm{l}\Lambda$ and $\partial_\mathrm{r}\Lambda$ are disjoint for $N > 1$. In the last product instead the shorten notation $b_{\mathrm{r/l},x}$ means: take $b_{\mathrm{r},x}$ if $x \in \partial_\mathrm{l}^{\mathrm{ext}}S_i$ only, take $b_{\mathrm{l},x}$ if $x \in \partial_\mathrm{r}^{\mathrm{ext}}S_i$ only, and take the product $b_{\mathrm{r},x}\, b_{\mathrm{l},x}$ in the case that x belongs to both $\partial_\mathrm{l}^{\mathrm{ext}}S_i$ and $\partial_\mathrm{r}^{\mathrm{ext}}S_j$.

Now substitute (2.10) into (2.3), using also the fact that $|\Lambda| = \sum_{i=1}^{n} |S_i| + \sum_{L\in\mathscr{L}_\Lambda(\cup_i S_i)} |L|$, and obtain:

$$
\begin{aligned}
Z_\Lambda^{\mathrm{h}} &= \lambda_1^{|\Lambda|} \Bigg(\prod_{x\in\partial_{\mathrm{v}}\Lambda} b_{\mathrm{l/r},\,x} \Bigg) \cdot \\
&\cdot \sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{S_1,\dots,S_n\,\in\,\mathscr{S}_\Lambda \\ \mathrm{dist}(S_i,S_j)>1\,\forall i\neq j}} \prod_{i=1}^{n} \left(\frac{e^{-\beta\left(\frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2}|S_i| + \frac{J}{2}|\partial_{\mathrm{h}} S_i|\right)}}{\lambda_1^{|S_i|}} \prod_{x\in\partial_{\mathrm{v}}\Lambda\cap\partial_{\mathrm{v}}S_i} \frac{e^{-\beta\frac{J}{2}}}{b_{\mathrm{l/r},\,x}} \right) \cdot \\
&\cdot \sum_{p\geq 0} \frac{1}{p!} \sum_{\substack{L_1,\dots,L_p\,\in\,\mathscr{L}_\Lambda(\cup_i S_i) \\ L_k\neq L_h\,\forall k\neq h}} \left(\prod_{k=1}^{p} R_{L_k} \right) \left(\prod_{\substack{x\in(\cup_i\partial_{\mathrm{v}}^{\mathrm{ext}}S_i)\cap\Lambda \\ x\notin\cup_k L_k}} b_{\mathrm{r/l},\,x} \right) .
\end{aligned}
\tag{2.11}
$$

The next step is to partition $\bigcup_{i=1}^{n} S_i \cup \bigcup_{k=1}^{p} L_k$ into connected components as a sub-graph of $\widetilde{\mathbb{Z}}^2$, where $\widetilde{\mathbb{Z}}^2$ is the lattice obtained from $\mathbb{Z}^2$ by removing all the vertical bonds incident to the lines L_k :

$$
\bigcup_{i=1}^{n} S_i \,\cup\, \bigcup_{k=1}^{p} L_k \;=\; \bigcup_{t=1}^{q} \mathrm{supp}\, P_t \quad ,
$$

$$
P_t \in \mathscr{P}_\Lambda \;\forall t \quad , \quad \mathrm{dist}_{\widetilde{\mathbb{Z}}^2}(\mathrm{supp}\, P_t, \mathrm{supp}\, P_s) > 1 \;\forall t \neq s
$$

where the family $\mathscr{P}_\Lambda$ (yes, it is finally our family of polymers! see fig.3) is defined by:

$$
\mathscr{P}_\Lambda := \left\{ P \equiv \big((S_i)_{i\in I}, (L_k)_{k\in K}\big) \;\middle|\; (S_i)_i \in \mathscr{P}\!\mathscr{S}_\Lambda \,,\; (L_k)_k \in \mathscr{P}\!\mathscr{L}_\Lambda(\cup_i S_i) \right\} \;, \tag{2.12}
$$

$$
(S_i)_{i\in I} \in \mathscr{P}\!\mathscr{S}_\Lambda \;\overset{\mathrm{def}}{\Leftrightarrow}\; \begin{cases} 0 \leq |I| < \infty \\ S_i \in \mathscr{S}_\Lambda \;\forall i \\ \mathrm{dist}_{\mathbb{Z}^2}(S_i, S_j) > 1 \;\forall i \neq j \;, \end{cases} \tag{2.13}
$$

$$
(L_k)_{k\in K} \in \mathscr{P}\!\mathscr{L}_\Lambda(\cup_{i\in I} S_i) \;\overset{\mathrm{def}}{\Leftrightarrow}\; \begin{cases} 0 \leq |K| < \infty,\; |I| + |K| \geq 1 \\ L_k \in \mathscr{L}_\Lambda(\cup_i S_i) \;\forall k \\ L_k \neq L_h \;\forall k \neq h \\ (\cup_i S_i) \cup (\cup_k L_k) \text{ connected in } \widetilde{\mathbb{Z}}^2 \;. \end{cases} \tag{2.14}
$$

The identity (2.11) now rewrites as

$$
Z_\Lambda^{\mathrm{h}} \;=\; C_\Lambda \sum_{q\geq 0} \frac{1}{q!} \sum_{P_1,\dots,P_q\in\mathscr{P}_\Lambda} \prod_{t=1}^{q} \varrho_\Lambda(P_t) \prod_{t<s} \delta(P_t, P_s) \tag{2.15}
$$

by setting, for all $P, P' \in \mathscr{P}_\Lambda$ with $P = \big((S_i)_{i\in I}, (L_k)_{k\in K}\big)$,

$$
C_\Lambda \;:=\; \lambda_1^{|\Lambda|} \prod_{x\in\partial_{\mathrm{v}}\Lambda} b_{\mathrm{l/r},\,x} \;, \tag{2.16}
$$

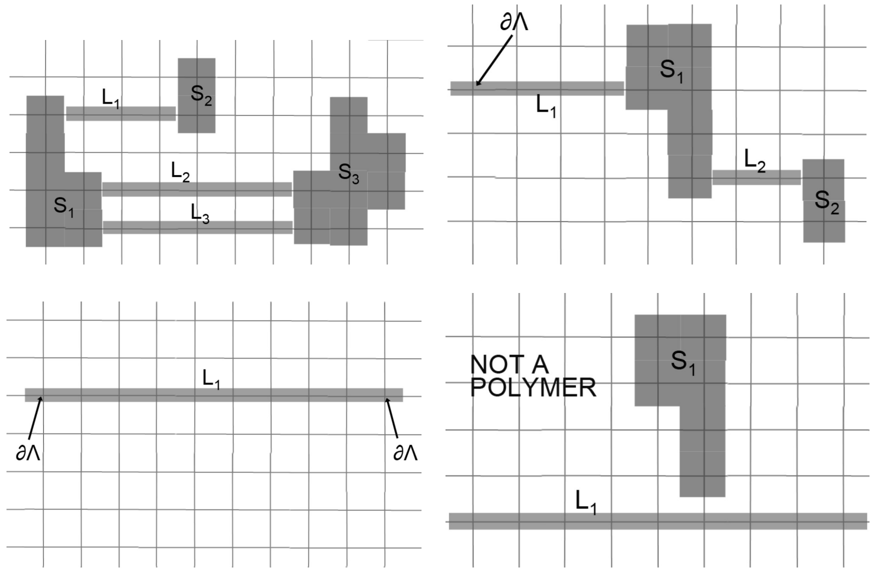

Fig. 3: The first three pictures represent three different examples of polymers $P \in \mathscr{P}_\Lambda$. The set represented in the last picture is not a unique polymer since it is not connected in $\widetilde{\mathbb{Z}}^2$ (even if it is connected in $\mathbb{Z}^2$).

$$\begin{aligned}\varrho_\Lambda(P) \;:=\; &\left(\frac{1}{|I|!}\prod_{i\in I}\left(\frac{e^{-\beta\left(\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}|S_i|+\frac{J}{2}|\partial_\mathrm{h}S_i|\right)}}{\lambda_1^{|S_i|}}\prod_{x\in\partial_\mathrm{v}\Lambda\cap\partial_\mathrm{v}S_i}\frac{e^{-\beta\frac{J}{2}}}{b_{\mathrm{l/r},\,x}}\right)\right)\cdot\\ &\cdot\left(\frac{1}{|K|!}\prod_{k\in K}R_{L_k}\right)\left(\prod_{\substack{x\in(\bigcup_{i\in I}\partial_\mathrm{v}^\mathrm{ext}S_i)\cap\Lambda\\ x\notin\bigcup_{k\in K}L_k}}b_{\mathrm{r/l},\,x}\right),\end{aligned} \tag{2.17}$$

$$\delta(P,P') \;:=\; \begin{cases}1\,, & \text{if } \operatorname{dist}_{\widetilde{\mathbb{Z}}^2}(P,P')>1\\ 0\,, & \text{otherwise}\end{cases}\;. \tag{2.18}$$

The identity (2.15) finally shows that the partition function Z_Λ^h, up to a factor C_Λ, admits a polymer representation of the form (B.1).

It is convenient to bound the polymer activity ϱ_Λ by a simpler quantity. Using the proposition A.9 plus the lemmas A.6, A.8 and the fact that $|\partial_\mathrm{h}S_i|\geq 2$, one finds:

$$\varrho_\Lambda(P)\;\leq\;\widetilde{\varrho}(P)\;:=\;\left(\frac{1}{|I|!}\prod_{i\in I}e^{-\beta\left(\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}|S_i|+J\right)}\right)\left(\frac{1}{|K|!}\prod_{k\in K}e^{-m|L_k|}\,\gamma_{L_k}\right) \tag{2.19}$$

with the γ_L's defined by the equation (A.9).

3 Convergence of the Cluster Expansion

In the previous section we rewrote our partition function Z_Λ^{h} as a polymer partition function up to a factor C_Λ (see formula (2.15)). In this section we will find a region of the parameters space $\mu_{\mathrm{h}}, \mu_{\mathrm{v}}, J$ where the condition (B.2) is verified by our model at low temperature, so that the general theorem B.1 about the convergence of the cluster expansion will apply to our case.

Theorem 3.1 *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. By choosing*

$$a(P) := \frac{m}{2}\,|\operatorname{supp} P| \quad \forall\, P \in \mathscr{P}_\Lambda \tag{3.1}$$

the conditions

$$\sum_{\substack{P \in \mathscr{P}_\Lambda \\ \operatorname{supp} P \ni x}} \widetilde{\varrho}(P)\, e^{a(P)} \;\le\; \frac{m}{8} \quad \forall\, x \in \Lambda\,, \tag{3.2}$$

$$\sum_{\substack{P \in \mathscr{P}_\Lambda \\ \delta(P,P^*)=0}} \widetilde{\varrho}(P)\, e^{a(P)} \;\le\; a(P^*) \quad \forall\, P^* \in \mathscr{P}_\Lambda \tag{3.3}$$

hold true, provided that $\beta > \beta_0$ and $N > N_0(\beta)$ (N is the minimum distance between two vertical components of $\partial\Lambda$). Here $\beta_0 > 0$ depends on $\mu_{\mathrm{h}}, \mu_{\mathrm{v}}, J$ only, while $N_0(\beta)$ depends on $\beta, \mu_{\mathrm{h}}, J$ only; they do not depend on Λ, P^*, x.

Corollary 3.2 *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. Suppose also that $\beta > \beta_0$ and $N > N_0(\beta)$. Denote by $\mathscr{C}\mathscr{P}_\Lambda$ the set of clusters*[13] *composed by polymers of $\mathscr{P}_\Lambda$. Then the partition function* (1.4) *rewrites as*

$$Z_\Lambda^{\mathrm{h}} \;=\; C_\Lambda \,\exp\Bigg(\sum_{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda}^{*} U_\Lambda\big((P_t)_t\big) \Bigg) \tag{3.4}$$

where we denote $\sum^*_{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda} := \sum_{q\ge 0} \frac{1}{q!} \sum_{(P_t)_{t=1}^q \in \mathscr{C}\mathscr{P}_\Lambda}$ and

$$U_\Lambda(P_1,\ldots,P_q) := u(P_1,\ldots,P_q) \prod_{t=1}^{q} \varrho_\Lambda(P_t)\,. \tag{3.5}$$

Remind that C_Λ is defined by (2.16)*, ϱ_Λ is defined by* (2.17) *and u is defined by* (B.4)*,* (2.18)*. Furthermore for all $\mathscr{E} \subseteq \mathscr{P}_\Lambda$ it holds*

$$\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t\colon P_t \in \mathscr{E}}}^{*} \big| U_\Lambda\big((P_t)_t\big) \big| \;\le\; \sum_{\substack{P \in \mathscr{P}_\Lambda \\ P \in \mathscr{E}}} |\varrho_\Lambda(P)|\, e^{a(P)} \tag{3.6}$$

where a is defined by (3.1).

[13] As explained in the Appendix B, using the definition (2.18) for δ, a family of polymers $(P_1,\ldots,P_q)$ is a *cluster* iff $\cup_{t=1}^q \operatorname{supp} P_t$ is connected in $\widetilde{\mathbb{Z}}^2$.

Proof The corollary follows from the general theory of cluster expansion (theorem B.1), since Z^{h}_Λ admits a polymer representation (2.15) and satisfies the Kotecky-Preiss condition ((3.3), $|\varrho_\Lambda| \le \widetilde{\varrho}$). $\square$

For ease of reading, in the following of this section we will denote

$$\sum_{(S_i)_i}^{*} := \sum_n \frac{1}{n!} \sum_{(S_i)_{i=1}^n \in \mathscr{P}\mathscr{S}_\Lambda} \qquad \text{and} \qquad \sum_{(L_k)_k}^{*} := \sum_p \frac{1}{p!} \sum_{(L_k)_{k=1}^p \in \mathscr{P}\mathscr{L}_\Lambda(\cup_i S_i)}$$

where $\mathscr{P}\mathscr{S}_\Lambda$, $\mathscr{P}\mathscr{L}_\Lambda(\cup_i S_i)$ are the projections of the polymer set P_Λ defined in (2.13), (2.14). The next lemmas provide the entropy estimates that will be needed in the proof of theorem 3.1.

Lemma 3.3 *If $\cup_i S_i \neq \emptyset$, namely $n \ge 1$, then*

$$\sum_{(L_k)_k}^{*} 1 \le 4^{\sum_i |S_i|} . \tag{3.7}$$

Proof Fix $p \ge 0$ and denote by $\mathscr{P}\mathscr{L}^{(p)}_\Lambda(\cup_i S_i)$ the set of $(L_k)_{k=1}^p \in \mathscr{P}\mathscr{L}_\Lambda(\cup_i S_i)$. Given $(L_k)_{k=1}^p \in \mathscr{P}\mathscr{L}^{(p)}_\Lambda(\cup_i S_i)$, each line L_k has at least one endpoint on $\cup_i \partial^{\mathrm{ext}}_{\mathrm{v}} S_i$, since $(\cup_i S_i) \cup (\cup_k L_k)$ have to be connected in $\widetilde{\mathbb{Z}}^2$. Therefore the number of ways to choose each L_k is at most $\sum_i |\partial^{\mathrm{ext}}_{\mathrm{v}} S_i| \le 2 \sum_i |S_i|$. Since the L_k, $k = 1, \dots, p$, must be all distinct, it follows that

$$\Big| \mathscr{P}\mathscr{L}^{(p)}_\Lambda(\cup_i S_i) \Big| \le \big(2 \sum\nolimits_i |S_i|\big) \big(2 \sum\nolimits_i |S_i| - 1\big) \cdots \big(2 \sum\nolimits_i |S_i| - p + 1\big) .$$

Therefore

$$\sum_{(L_k)_k}^{*} 1 = \sum_p \frac{1}{p!} \Big| \mathscr{P}\mathscr{L}^{(p)}_\Lambda(\cup_i S_i) \Big| \le \sum_p \binom{2 \sum_i |S_i|}{p} = 2^{2 \sum_i |S_i|} .$$

$\square$

Lemma 3.4 *Let $x \in \mathbb{Z}^2$. For all $s \ge 2$*

$$\#\big\{ S \subset \mathbb{Z}^2 \text{ connected} \,\big|\, |S| = s\,,\ S \ni x \big\} \le \frac{16}{3}\, 4^{4s} . \tag{3.8}$$

Proof Given a connected graph G and one of its vertices x, there exists a walk in G that starts from x and crosses each edge exactly twice[14]. Therefore

$$\begin{aligned}
&\#\big\{ S \subset \mathbb{Z}^2 \text{ connected} \,\big|\, |S| = s\,,\ S \ni x \big\} \le \\
&\le \sum_{e=s-1}^{2s} \#\big\{ S \text{ connected sub-graph of } \mathbb{Z}^2 \,\big|\, |\text{edges of } S| = e\,,\ S \ni x \big\} \\
&\le \sum_{e=s-1}^{2s} \#\big\{ \text{walks in } \mathbb{Z}^2 \text{ that start from } x \text{ and have lenght } 2e \big\} \\
&\le \sum_{e=s-1}^{2s} 4^{2e} \le \frac{4^{4s+2}}{3} .
\end{aligned}$$

$\square$

[14] This can be easily proven by induction on the number of edges.

Lemma 3.5 *Let $A \subset \mathbb{Z}^2$ finite. For all $s \geq 2$, $1 \leq d < \infty$*

$$\#\big\{S \subset \mathbb{Z}^2 \text{ connected} \,\big|\, |S| = s\,,\ \mathrm{dist_h}(S,A) = d\big\} \ \leq\ \frac{32}{3}\,|A|\,4^{4s}\,. \tag{3.9}$$

Here $\mathrm{dist_h}(S,A) := \inf_{x\in S,\, y\in A} \mathrm{dist_h}(x,y)$ *and the horizontal distance between* $x = (x_\mathrm{h}, x_\mathrm{v})$, $y = (y_\mathrm{h}, y_\mathrm{v}) \in \mathbb{Z}^2$ *is defined as*

$$\mathrm{dist_h}(x,y) \ :=\ \begin{cases} |x_\mathrm{h} - y_\mathrm{h}| & \text{if } x_\mathrm{v} = y_\mathrm{v} \\ +\infty & \text{if } x_\mathrm{v} \neq y_\mathrm{v} \end{cases}. \tag{3.10}$$

Proof Observe that $\mathrm{dist_h}(S,A) = d$ if and only if there exists a horizontal line L, $|L| = d+1$, having one endpoint on $\partial_\mathrm{v} A$ and the other one on $\partial_\mathrm{v} S$. Therefore:

$$\begin{aligned} &\#\big\{S \subset \mathbb{Z}^2 \text{ connected} \,\big|\, |S| = s\,,\ \mathrm{dist_h}(S,A) = d\big\} \ \leq \\ &\leq \sum_{\substack{L \text{ horiz. line, } |L|=d+1, \\ \partial_\mathrm{v} A \ni \text{ one endpt. of } L}} \#\big\{S \subset \mathbb{Z}^2 \text{ connected} \,\big|\, |S| = s\,,\ \partial_\mathrm{v} S \ni \text{other endpt. of } L\big\} \\ &\leq\ 2|\partial_\mathrm{v} A|\ \#\big\{S \subset \mathbb{Z}^2 \text{ connected} \,\big|\, |S| = s\,,\ S \ni 0\big\} \ \leq\ 2|A|\ \frac{16}{3}\,4^{4s}\,. \end{aligned}$$

For the last inequality we have used the lemma 3.4. □

Lemma 3.6 *Let $n \geq 1$. Let $\mathcal{T}$ be a tree over the vertices $\{1,\ldots,n\}$. Let $s_i \geq 2$ for all $i = 1,\ldots,n$ and $d_{ij} \geq 2$ for all $(i,j) \in \mathcal{T}$.*
Then given $A \subset \mathbb{Z}^2$ and $1 \leq d < \infty$

$$\begin{aligned} &\#\big\{(S_i)_{i=1}^n \in \mathscr{PS}_A \,\big|\, \mathrm{dist_h}(S_1,A) = d\,,\ |S_i| = s_i\,\forall i\,, \\ &\qquad\qquad \mathrm{dist_h}(S_i,S_j) = d_{ij}\,\forall (i,j)\in\mathcal{T}\big\} \ \leq \\ &\leq\ |A| \prod_{i=1}^n \left(\frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\ ; \end{aligned} \tag{3.11}$$

while given $x \in \mathbb{Z}^2$

$$\begin{aligned} &\#\big\{(S_i)_{i=1}^n \in \mathscr{PS}_A \,\big|\, S_1 \ni x\,,\ |S_i| = s_i\,\forall i\,, \\ &\qquad\qquad \mathrm{dist_h}(S_i,S_j) = d_{ij}\,\forall (i,j)\in\mathcal{T}\big\} \ \leq \\ &\leq\ \prod_{i=1}^n \left(\frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\ . \end{aligned} \tag{3.12}$$

Here $\deg_\mathcal{T}(i)$ denotes the degree of the vertex i in the tree $\mathcal{T}$.

Proof Let start by proving the inequality (3.11) by induction on n. If $n = 1$, then the tree $\mathcal{T}$ is trivial and (3.11) is already provided by the lemma 3.5. Now let $n \geq 2$, assume that (3.11) holds for at most $n-1$ vertices and prove it for n. It is convenient to think that the tree $\mathcal{T}$ is rooted at the vertex 1 and denote by $j \leftarrow i$ the relation "vertex j is son of vertex i in $\mathcal{T}$" and by $\mathcal{T}(i)$ the sub-tree of $\mathcal{T}$ induced by the vertex i together with its descendants. Then,

denoting by $N_{\mathcal{T},1}\big(A,d;(s_i)_{i\in\mathcal{T}},(d_{ij})_{(i,j)\in\mathcal{T}}\big)$ the cardinality on the l.h.s. of (3.11), it holds

$$N_{\mathcal{T},1}\big(A,d;(s_i)_{i\in\mathcal{T}},(d_{ij})_{(i,j)\in\mathcal{T}}\big) = \\ = \sum_{\substack{S_1\in\mathscr{S}_\Lambda,\ |S_1|=s_1\\ \mathrm{dist_h}(S_1,A)=d}} \prod_{v\leftarrow 1} N_{\mathcal{T}(v),v}\big(S_1,d_{1v};(s_i)_{i\in\mathcal{T}(v)},(d_{ij})_{(i,j)\in\mathcal{T}(v)}\big)\ .$$

Since $\mathcal{T}(v)$ has at most $n-1$ vertices, the induction hypothesis gives

$$N_{\mathcal{T}(v),v}\big(S_1,d_{1v};(s_i)_{i\in\mathcal{T}(v)},(d_{ij})_{(i,j)\in\mathcal{T}(v)}\big) \le s_1 \prod_{i\in\mathcal{T}(v)} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}(v)}(i)}\right)\ .$$

Then by substituting in the previous identity, bounding $\deg_{\mathcal{T}(v)}(i)$ by $\deg_{\mathcal{T}}(i)$ and using the lemma 3.5, one obtains:

$$N_{\mathcal{T},1}\big(A,d;(s_i)_{i\in\mathcal{T}},(d_{ij})_{(i,j)\in\mathcal{T}}\big) \le |A| \prod_{i\in\mathcal{T}} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}}(i)}\right)\ .$$

This concludes the proof of (3.11).
In order to prove the inequality (3.12), denote by $N'_{\mathcal{T},1}\big(x;(s_i)_{i\in\mathcal{T}},(d_{ij})_{(i,j)\in\mathcal{T}}\big)$ the cardinality on the l.h.s. of (3.12) and observe that

$$N'_{\mathcal{T},1}\big(x;(s_i)_{i\in\mathcal{T}},(d_{ij})_{(i,j)\in\mathcal{T}}\big) = \\ \sum_{\substack{S_1\in\mathscr{S}_\Lambda,\ |S_1|=s_1\\ S_1\ni x}} \prod_{v\leftarrow 1} N_{\mathcal{T}(v),v}\big(S_1,d_{1v};(s_i)_{i\in\mathcal{T}(v)},(d_{ij})_{(i,j)\in\mathcal{T}(v)}\big)\ .$$

By (3.11) we already know that

$$N_{\mathcal{T}(v),v}\big(S_1,d_{1v};(s_i)_{i\in\mathcal{T}(v)},(d_{ij})_{(i,j)\in\mathcal{T}(v)}\big) \le s_1 \prod_{i\in\mathcal{T}(v)} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}(v)}(i)}\right)\ .$$

Then by substituting in the previous identity, bounding $\deg_{\mathcal{T}(v)}(i)$ by $\deg_{\mathcal{T}}(i)$ and using the lemma 3.4, one obtains:

$$N'_{\mathcal{T},1}\big(x;(s_i)_{i\in\mathcal{T}},(d_{ij})_{(i,j)\in\mathcal{T}}\big) \le \prod_{i\in\mathcal{T}} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}}(i)}\right)\ ,$$

which proves (3.12). □

Proof (of the theorem 3.1) According to the definition (2.18), the condition $\delta(P,P^*)=0$ implies that $\operatorname{supp} P\cap[\operatorname{supp} P^*]_1 \neq \emptyset$, where $[A]_1 := \{x\in\mathbb{Z}^2 \mid \mathrm{dist}_{\mathbb{Z}^2}(x,A)\le 1\}$. Therefore

$$\begin{aligned}\sum_{\substack{P\in\mathscr{P}_\Lambda\\ \delta(P,P^*)=0}} \widetilde{\varrho}(P)\, e^{a(P)} &\le \sum_{x\in[\operatorname{supp} P^*]_1} \sum_{\substack{P\in\mathscr{P}_\Lambda\\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)} \\ &\le 4\,|\operatorname{supp} P^*| \max_{x\in\Lambda} \sum_{\substack{P\in\mathscr{P}_\Lambda\\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)}\ .\end{aligned}$$

Thus, by choosing $a(P) := \frac{m}{2}|\operatorname{supp} P|$ for all $P \in \mathscr{P}_\Lambda$, the inequality (3.3) will be a consequence of (3.2).

We have to prove the inequality (3.2). It is worth to write down explicitly the quantity we will work with (see the definitions (2.19) and (3.1)):

$$\widetilde{\varrho}(P)\, e^{a(P)} \;=\; \left(\frac{1}{n!}\prod_{i=1}^{n} e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right)\left(\frac{1}{p!}\prod_{k=1}^{p} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}\right)$$

for all $P \in \mathscr{P}_\Lambda$, $P = \big((S_i)_{i=1}^n, (L_k)_{k=1}^p\big)$. Notice that if $\operatorname{supp} P \ni x$, the site x may belong either to a region S_i or to a line L_k; hence we can split the sum on the l.h.s. of (3.2) into two parts:

$$\sum_{\substack{P\in\mathscr{P}_\Lambda\\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)} \;=\; \Sigma_1 \,+\, \Sigma_2 \tag{3.13}$$

with

$$\Sigma_1 \;:=\; \sideset{}{^*}\sum_{\substack{(S_i)_i\\ \cup_i S_i\ni x}} \left(\prod_i e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right) \sideset{}{^*}\sum_{(L_k)_k} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \tag{3.14}$$

$$\Sigma_2 \;:=\; \sideset{}{^*}\sum_{(S_i)_i} \left(\prod_i e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right) \sideset{}{^*}\sum_{\substack{(L_k)_k\\ \cup_k L_k\ni x}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}\;. \tag{3.15}$$

During all the proof $o(1)$ will denote any function $\omega = \omega(\beta, \mu_{\rm h}, J)$ such that $\omega \to 0$ as $\beta \to \infty$ and ω depends only on β, $\mu_{\rm h}$, J (in particular it does not depend on the choices of $\Lambda \subset \mathbb{Z}^2$, $x \in \mathbb{Z}^2$, $P \in \mathscr{P}_\Lambda$).

I. *Study of the term Σ_1.*

We fix a family of regions $(S_i)_{i=1}^n$ that contains the point x; we also assume that $\mathscr{PL}_\Lambda(\cup_i S_i)$ is non-empty, otherwise the contribution to Σ_1 is zero. By the lemma 3.3 it holds

$$\sideset{}{^*}\sum_{(L_k)_k} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\le\; 4^{\sum_i |S_i|} \max_{(L_k)_k} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \tag{3.16}$$

where the maximum is taken over all $(L_k)_k \in \mathscr{PL}_\Lambda(\cup_i S_i)$. The factor γ_{L_k} can take two values (see formula (A.9)), both smaller than 1 for β sufficiently large (uniformly with respect to L_k), since each line L_k must have at least one endpoint on $\cup_i \partial_{\rm v}^{\rm ext} S_i$ to ensure that $(\cup_i S_i)\cup(\cup_k L_k)$ is connected in $\widetilde{\mathbb{Z}}^2$.

Obviously $n \ge 1$ in order for $\cup_{i=1}^n S_i$ to contain the point x. It is convenient to consider separately the case $n = 1$ and the case $n \ge 2$:

$$\Sigma_1 \;=\; \Sigma_1' + \Sigma_1''\;.$$

The case $n = 1$ is easy to deal with, simply by bounding the r.h.s. of (3.16) by $4^{|S|}$ and using the lemma 3.4. Precisely:

$$\begin{aligned}
\Sigma_1' &:= \sum_{\substack{S\in\mathscr{S}_\Lambda \\ S\ni x}} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S|-\beta J} \sum_{(L_k)_k}^{*} \prod_k e^{-\frac{m}{2}|L_k|}\gamma_{L_k} \\
&\le \sum_{\substack{S\in\mathscr{S}_\Lambda \\ S\ni x}} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S|-\beta J}\, 4^{|S|} \\
&\le \sum_{\substack{s\ge 2 \\ \text{even}}} \frac{16}{3}\, 4^{4s}\, e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)s-\beta J}\, 4^{s} \\
&= \frac{16}{3}\, 4^{10}\, e^{-\beta\,(\mu_\mathrm{h}-\mu_\mathrm{v}+J)}\,(1+o(1))\,.
\end{aligned} \tag{3.17}$$

Now assume $n \ge 2$. Fix a family of lines $(L_k)_{k=1}^{p} \in \mathscr{PL}_\Lambda(\cup_i S_i)$. We can consider the graph $G \equiv G\big((S_i)_i,(L_k)_k\big)$ with vertices $i \in \{1,\dots,n\}$ and edges $k \in \{1,\dots,p\}$: the edge k joins the two vertices i, j iff the line L_k has one endpoint on $\partial_\mathrm{v}^\mathrm{ext} S_i$ and the other one on $\partial_\mathrm{v}^\mathrm{ext} S_j$. In the graph G there can be multiple edges, loops and pseudo-edges with a single endpoint. The graph G is connected (it follows from definition 2.14), hence G admits at least one spanning sub-tree $\mathcal{T}$. And clearly, since each factor $e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}$ is smaller than 1,

$$\prod_{k=1}^{p} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \le \prod_{k\in\mathcal{T}} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \le \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}$$

where $\gamma_{S,S'} := \left(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}(\mathrm{dist_h}(S,S')-1)}\right)(1+o(1))$. Therefore:

$$\max_{(L_k)_k} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \le \max_{\substack{\mathcal{T}\text{ tree over} \\ \{1,\dots,n\}}} \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j} \tag{3.18}$$

Now using (3.16) and (3.18) we can bound Σ_1'':

$$\begin{aligned}
\Sigma_1'' &:= \sum_{n\ge 2}\frac{1}{n!}\sum_{\substack{(S_i)_{i=1}^n \\ \cup_i S_i\ni x}} \left(\prod_{i=1}^{n} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right) \sum_{(L_k)_k}^{*}\prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \\
&\le \sum_{n\ge 2}\sum_{\substack{\mathcal{T}\text{ tree over} \\ \{1,\dots,n\}}} \frac{1}{n!}\sum_{\substack{(S_i)_{i=1}^n \\ \cup_i S_i\ni x}} \left(\prod_{i=1}^{n} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)|S_i|-\beta J}\right)\cdot \\
&\qquad\qquad \cdot \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}
\end{aligned} \tag{3.19}$$

where in the sums we keep implicit that $(S_i)_{i=1}^{n} \in \mathscr{PS}_\Lambda$.

Substitute into (3.19) the entropy bound[15] (3.12). Since $\cup_i S_i \ni x$, but not necessarily $S_1 \ni x$, an extra factor n appears. Moreover observe that $|S_i|$ is even and ≥ 2 (see the definition (2.1)) and $\mathrm{dist}_{\mathrm{h}}(S_i, S_j) \geq 2$. Then:

$$\begin{aligned} \varSigma_1'' \leq & \sum_{n\geq 2} \sum_{\substack{\mathcal{T}\text{ tree over}\\ \{1,\dots,n\}}} \frac{n}{n!} \sum_{\substack{(s_i)_{i=1,\dots,n}\\ s_i \text{ even } \geq 2}} \sum_{\substack{(d_{ij})_{ij\in\mathcal{T}}\\ d_{ij}\geq 2}} \left(\prod_{i=1}^{n} \frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}}(i)} \right) \cdot \\ & \cdot \left(\prod_{i=1}^{n} e^{-\left(\beta \frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2} - \frac{m}{2} - \log 4\right) s_i - \beta J} \right) \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(d_{ij}-1)}\, \gamma_{d_{ij}} \end{aligned} \tag{3.20}$$

where $\gamma_d := \left(\frac{1}{2} e^{-\beta J} + e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}(d-1)}\right)(1+o(1))$.

Given $n \geq 2$ and $\delta_1, \dots, \delta_n \geq 1$, the number of trees $\mathcal{T}$ over the vertices $\{1,\dots,n\}$ with given degrees $\deg_{\mathcal{T}}(i) = \delta_i \ \ \forall i = 1,\dots,n$ is exactly[16]

$$\frac{(n-2)!}{(\delta_1-1)!\cdots(\delta_n-1)!}$$

if $\sum_{i=1}^{n}(\delta_i - 1) = n-2$ and zero otherwise. Furthermore the number of edges of $\mathcal{T}$ is $n-1$. Therefore the bound (3.20) leads to

$$\begin{aligned} \varSigma_1'' \leq & \sum_{n\geq 2} \left(\frac{32}{3}\, e^{-\beta J} \sum_{\substack{s\geq 2\\ \text{even}}} e^{-\left(\beta \frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2} - \frac{m}{2} - 5\log 4\right)s} \sum_{\delta\geq 1} \frac{s^{\delta}}{(\delta-1)!} \right)^{n} \cdot \\ & \cdot \left(\sum_{d\geq 2} e^{-\frac{m}{2}(d-1)}\, \gamma_d \right)^{n-1}. \end{aligned} \tag{3.21}$$

The sum over s gives, as $\beta \to \infty$,

$$\begin{aligned} & \sum_{\substack{s\geq 2\\ \text{even}}} e^{-\left(\beta \frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2} - \frac{m}{2} - 5\log 4\right)s} \sum_{\delta\geq 1} \frac{s^{\delta}}{(\delta-1)!} = \\ & = \sum_{\substack{s\geq 2\\ \text{even}}} s\, e^{-\left(\beta \frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2} - \frac{m}{2} - 5\log 4 - 1\right)s} = 2\, e^2\, 4^{10}\, e^{-\beta(\mu_{\mathrm{h}}-\mu_{\mathrm{v}})}\,(1+o(1)). \end{aligned} \tag{3.22}$$

The sum over d gives, as $\beta \to \infty$,

$$\begin{aligned} & \sum_{d\geq 2} e^{-\frac{m}{2}(d-1)}\, \gamma_d = \\ & = \left(\sum_{d\geq 2} e^{-\frac{m}{2}(d-1)}\, \frac{e^{-\beta J}}{2} + \sum_{d\geq 2} e^{-\frac{m}{2}(d-1)}\, e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}(d-1)} \right)(1+o(1)) \\ & = \left(\frac{1}{1-e^{-\frac{m}{2}}}\, \frac{e^{-\beta J}}{2} + o(1) \right)(1+o(1)) = e^{\beta \frac{\mu_{\mathrm{h}}+J}{2}}\,(1+o(1)) \end{aligned} \tag{3.23}$$

[15] The families of regions $(S_i)_{i=1}^{n}$ such that $\mathrm{dist}_{\mathrm{h}}(S_i, S_j) = \infty$ for at least one edge $(i,j) \in \mathcal{T}$ give zero contribution to the sum, therefore we do not need to worry about them.

[16] This is an improvement of the well-known Cayley's formula.

where we used the fact that $1 - e^{-\frac{m}{2}} = \frac{1}{2}\, e^{-\beta\frac{\mu_{\rm h}+3J}{2}}\,(1+o(1))$ (see lemma A.5). Substituting (3.22), (3.23) into (3.21), one obtains

$$\varSigma_1'' \le \sum_{n\ge 2} \left(\frac{2^{26}e^2}{3}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+J}{2}}\,(1+o(1)) \right)^n e^{-\beta\frac{\mu_{\rm h}+J}{2}}\,(1+o(1))\,. \tag{3.24}$$

Assume $\boldsymbol{\mu}_{\rm h} - \boldsymbol{\mu}_{\rm v} > \frac{\boldsymbol{\mu}_{\rm h}+\boldsymbol{J}}{\boldsymbol{2}}$. Then for β sufficiently large (3.24) becomes:

$$\begin{aligned} \varSigma_1'' &\le \left(\frac{2^{26}e^2}{3}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+J}{2}} \right)^2 e^{-\beta\frac{\mu_{\rm h}+J}{2}}\,(1+o(1)) \\ &= \frac{2^{52}e^4}{9}\, e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\,\frac{\mu_{\rm h}+J}{2}}\,(1+o(1))\,. \end{aligned} \tag{3.25}$$

II. *Study of the term $\varSigma_2$.*

The ideas are not far from those already seen for $\varSigma_1$. We fix a family of regions $(S_i)_{i=1}^n$ and we assume that there exists $(L_k)_k \in \mathscr{PL}_\varLambda(\cup_i S_i)$ such that $\cup_k L_k \ni x$, otherwise the contribution to $\varSigma_2$ is zero. Clearly the line $L^x \in \mathscr{L}_\varLambda(\cup_i S_i)$ that contains x is unique. It is convenient to consider separately four cases:

$$\varSigma_2 = \varSigma_2' + \varSigma_2'' + \varSigma_2''' + \varSigma_2''''\,.$$

In $\varSigma_2'$ we assume $n = 0$, namely $\cup_i S_i = \emptyset$; then L^x have to be a maximal horizontal line of $\varLambda$. In $\varSigma_2''$ we assume $n = 1$, namely there is a unique region S and L^x may have one endpoint on $\partial_{\rm v}^{\rm ext} S$ and one on $\partial_{\rm v}\varLambda$ or both on $\partial_{\rm v}^{\rm ext} S$. In $\varSigma_2'''$ we assume $n \ge 2$ and L^x has one endpoint on $\cup_i \partial_{\rm v}^{\rm ext} S_i$ and one on $\partial_{\rm v}\varLambda$ or both on the same $\partial_{\rm v}^{\rm ext} S_i$. In $\varSigma_2''''$ we assume $n \ge 2$ and L^x has one endpoint on $\partial_{\rm v}^{\rm ext} S_i$ and one on $\partial_{\rm v}^{\rm ext} S_j$ with $i \ne j$.

The case $n = 0$ is easy to deal with. Indeed, since the unique $(L_k)_k \in \mathscr{PL}_\varLambda(\emptyset)$ such that $\cup_k L_k \ni x$ is the singleton (L^x),

$$\varSigma_2' := \sideset{}{^*}\sum_{\substack{(L_k)_k \\ \cup_k L_k \ni x}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} = e^{-\frac{m}{2}|L^x|}\,\gamma_{L^x} \le e^{-\frac{m}{2}N}\,(1+o(1)) \tag{3.26}$$

where N denotes the minimum distance between two different vertical components of $\partial\varLambda$.

When $n \ge 1$, by the lemma 3.3 it holds:

$$\sideset{}{^*}\sum_{\substack{(L_k)_k \\ \cup_k L_k \ni x}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \le 4^{\sum_i |S_i|} \max_{\substack{(L_k)_k \\ \cup_k L_k \ni x}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \tag{3.27}$$

where it is implicit in the notation that $(L_k)_k \in \mathscr{PL}_\varLambda(\cup_i S_i)$. The factor γ_{L_k} can take two values (see formula (A.9)), both smaller than 1 for β sufficiently large (uniformly with respect to L_k), since each line L_k must have at least one endpoint on $\cup_i \partial_{\rm v}^{\rm ext} S_i$.

Now the case $n = 1$ is also easy to deal with. Indeed, by bounding the r.h.s. of (3.27) by $4^{|S|}\, e^{-\frac{m}{2}|L^x|}\,\gamma_{L^x}$, one obtains:

$$\begin{aligned} \Sigma_2'' &:= \sum_{S\in\mathscr{S}_\Lambda} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S|} \sideset{}{^*}\sum_{\substack{(L_k)_k\\ \cup_k L_k\ni x}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \\ &\le \sum_{S\in\mathscr{S}_\Lambda} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S|}\; 4^{|S|}\, e^{-\frac{m}{2}|L^x|}\,\gamma_{L^x}\ ; \end{aligned} \tag{3.28}$$

then observe that $|L^x| \ge \mathrm{dist}_\mathrm{h}(S,x)$ and use the lemma 3.5 for the entropy:

$$\Sigma_2'' \le \sum_{\substack{s\ge 2\\ \text{even}}} \sum_{d\ge 1} \frac{32}{3}\, 4^{4s}\, e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)s}\, 4^s\, e^{-\frac{m}{2}d}\,\gamma_d \tag{3.29}$$

where $\gamma_d := \left(\frac{e^{-\beta\frac{J}{2}}}{\sqrt{2}} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}d}\right)(1+o(1))$; finally compute the geometric series in s, d, and use $1 - e^{-\frac{m}{2}} = \frac{1}{2}e^{-\beta\frac{\mu_\mathrm{h}+3J}{2}}\,(1+o(1))$ (see lemma A.5) to obtain:

$$\begin{aligned} \Sigma_2'' &\le \frac{32}{3}\left(4^{10}\, e^{-\beta(\mu_\mathrm{h}-\mu_\mathrm{v})}\right)\left(\frac{1}{1-e^{-\frac{m}{2}}}\,\frac{e^{-\beta\frac{J}{2}}}{\sqrt{2}} + o(1)\right)(1+o(1)) \\ &= \frac{2^{25}\sqrt{2}}{3}\, e^{-\beta\,(\mu_\mathrm{h}-\mu_\mathrm{v})+\beta\,\frac{\mu_\mathrm{h}+2J}{2}}\,(1+o(1))\ . \end{aligned} \tag{3.30}$$

Assume now $n \ge 2$ and that L^x has one endpoint on $\cup_i\partial_\mathrm{v}^\mathrm{ext}S_i$ and the other one on $\partial^\mathrm{ext}\Lambda$ or both endpoints on the same $\partial_\mathrm{v}^\mathrm{ext}S_i$. By introducing an extra factor n we may assume that one endpoint is on $\partial_\mathrm{v}^\mathrm{ext}S_1$. Fix a family of lines $(L_k)_{k=1}^p \in \mathscr{PL}_\Lambda(\cup_i S_i)$ such that $\cup_k L_k \ni x$ (namely there is a k such that $L_k = L^x$). We can introduce the graph $G \equiv G\big((S_i)_i,(L_k)_k\big)$ with vertices $i \in \{1,\ldots,n\}$ and edges $k \in \{1,\ldots,p\}$: the edge k joins the two vertices i, j iff the line L_k has one endpoint on $\partial_\mathrm{v}^\mathrm{ext}S_i$ and the other one on $\partial_\mathrm{v}^\mathrm{ext}S_j$. The graph G is connected, hence G admits at least one spanning sub-tree $\mathcal{T}$. Notice that the line L^x is not part of this tree. Hence, since each factor $e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}$ is smaller than 1,

$$\begin{aligned} \prod_{k=1}^p e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} &\le e^{-\frac{m}{2}|L^x|}\,\gamma_{L^x} \prod_{k\in\mathcal{T}} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \\ &\le e^{-\frac{m}{2}\,\mathrm{dist}_\mathrm{h}(S_1,x)}\,\gamma_{S_1,x} \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist}_\mathrm{h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j} \end{aligned}$$

where $\gamma_{S,x} := \left(\frac{1}{\sqrt{2}}e^{-\beta\frac{J}{2}} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}\,\mathrm{dist_h}(S,x)}\right)(1+o(1))$ and $\gamma_{S,S'} := \left(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}(\mathrm{dist_h}(S,S')-1)}\right)(1+o(1))$. Therefore:

$$\begin{aligned}
&\max_{\substack{(L_k)_k,\,\cup_k L_k \ni x,\\ L^x \text{ from } \partial_\mathrm{v}^\mathrm{ext} S_1 \text{ to } \partial_\mathrm{v}\Lambda\\ \text{or from a } \partial_\mathrm{v}^\mathrm{ext} S_1 \text{ to itself}}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \le \\
&\le e^{-\frac{m}{2}\,\mathrm{dist_h}(S_1,x)}\,\gamma_{S_1,x} \max_{\substack{\mathcal{T} \text{ tree over}\\ \{1,\dots,n\}}} \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}\,.
\end{aligned} \tag{3.31}$$

Now using (3.27) and (3.31) we can bound Σ_2''':

$$\begin{aligned}
\Sigma_2''' &:= \sum_{n\ge 2}\frac{1}{n!}\sum_{(S_i)_{i=1}^n}\left(\prod_i e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right)\cdot \\
&\qquad\qquad \cdot \sum^*_{\substack{(L_k)_k,\,\cup_k L_k\ni x,\\ L^x \text{ from } \cup_i\partial_\mathrm{v}^\mathrm{ext}S_i \text{ to } \partial_\mathrm{v}\Lambda\\ \text{or from a } \partial_\mathrm{v}^\mathrm{ext}S_i \text{ to itself}}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \\
&\le \sum_{n\ge 2}\sum_{\substack{\mathcal{T} \text{ tree over}\\ \{1,\dots,n\}}}\frac{n}{n!}\sum_{(S_i)_{i=1}^n}\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)|S_i|-\beta J}\right)\cdot \\
&\qquad\qquad \cdot e^{-\frac{m}{2}\,\mathrm{dist_h}(S_1,x)}\,\gamma_{S_1,x}\prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}\,.
\end{aligned} \tag{3.32}$$

Substitute into (3.32) the entropy bound (3.11):

$$\begin{aligned}
\Sigma_2''' &\le \sum_{n\ge 2}\sum_{\substack{\mathcal{T} \text{ tree over}\\ \{1,\dots,n\}}}\frac{n}{n!}\sum_{\substack{(s_i)_{i=1,\dots,n}\\ s_i \text{ even } \ge 2}}\sum_{d_*\ge 1}\sum_{\substack{(d_{ij})_{(i,j)\in\mathcal{T}}\\ d_{ij}\ge 2}}\left(\prod_{i=1}^n \frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\cdot \\
&\quad \cdot\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)s_i-\beta J}\right) e^{-\frac{m}{2}d_*}\,\bar\gamma_{d_*}\prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(d_{ij}-1)}\,\gamma_{d_{ij}}
\end{aligned} \tag{3.33}$$

where $\gamma_d := \left(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}(d-1)}\right)(1+o(1))$ and $\bar\gamma_d := \left(\frac{1}{\sqrt{2}}e^{-\beta\frac{J}{2}} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}d}\right)(1+o(1))$. Observe that (3.33) is identical to (3.20) up to an extra factor $\sum_{d_*\ge 1} e^{-\frac{m}{2}d_*}\,\bar\gamma_{d_*}$, which equals

$$\sum_{d_*\ge 1} e^{-\frac{m}{2}d_*}\,\bar\gamma_{d_*} = \left(\frac{1}{1-e^{-\frac{m}{2}}}\,\frac{e^{-\beta\frac{J}{2}}}{\sqrt{2}} + o(1)\right)(1+o(1)) = \sqrt{2}\,e^{\beta\frac{\mu_\mathrm{h}+2J}{2}}\,(1+o(1))$$

since $1-e^{-\frac{m}{2}} = \frac{1}{2}\, e^{-\beta\frac{\mu_{\rm h}+3J}{2}}\,(1+o(1))$. Therefore we assume $\boldsymbol{\mu}_{\rm h} - \boldsymbol{\mu}_{\rm v} > \frac{\boldsymbol{\mu}_{\rm h}+\boldsymbol{J}}{\boldsymbol{2}}$ and we exploit the inequality (3.25) to bound the expression (3.33):

$$\begin{aligned} \Sigma_2''' \;&\le\; \left(\frac{2^{26}e^2}{3}\right)^2 e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+J}{2}}\;\sqrt{2}\, e^{\beta\frac{\mu_{\rm h}+2J}{2}}\,(1+o(1)) \\ &=\; \frac{2^{52}e^4\sqrt{2}}{9}\; e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{2\mu_{\rm h}+3J}{2}}\,(1+o(1))\;. \end{aligned} \tag{3.34}$$

Finally assume $n \ge 2$ and that L^x has one endpoint on $\partial_{\rm v}^{\rm ext} S_i$ and one on $\partial_{\rm v}^{\rm ext} S_j$ with $i \ne j$. By introducing an extra factor $n(n-1)/2$ we may assume that these endpoints lie on $\partial_{\rm v}^{\rm ext} S_1$ and $\partial_{\rm v}^{\rm ext} S_2$ respectively. Fix a family of lines $(L_k)_{k=1}^p \in \mathscr{PL}_\Lambda(\cup_i S_i)$ such that $\cup_k L_k \ni x$ (namely there exists $k \equiv k_x$ such that $L_k = L^x$), then consider the graph $G \equiv G\big((S_i)_i, (L_k)_k\big)$ with vertices $i \in \{1,\dots,n\}$ and edges $k \in \{1,\dots,p\}$: the edge k joins the two vertices i,j if the line L_k has one endpoint on $\partial_{\rm v}^{\rm ext} S_i$ and the other one on $\partial_{\rm v}^{\rm ext} S_j$. G admits at least one spanning sub-tree $\mathcal{T}$ that includes the edge k_x . Therefore

$$\begin{aligned} &\prod_{k=1}^p e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\le\; \prod_{k\in\mathcal{T}} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\le \\ &\le\; e^{-\frac{m}{2}({\rm dist_h}(S_1,x)+{\rm dist_h}(S_2,x)-1)}\,\bar\gamma_{S_1,S_2,x} \prod_{\substack{(i,j)\in\mathcal{T}\\ (i,j)\ne(1,2)}} e^{-\frac{m}{2}({\rm dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j} \end{aligned}$$

where $\bar\gamma_{S,S',x} := \big(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_{\rm h}+J}{2}\left({\rm dist_h}(S,x)+{\rm dist_h}(S',x)-1\right)}\big)\,(1+o(1))$ and $\gamma_{S,S'} := \big(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_{\rm h}+J}{2}\left({\rm dist_h}(S,S')-1\right)}\big)\,(1+o(1))$. Thus:

$$\begin{aligned} &\max_{\substack{(L_k)_k,\,\cup_k L_k\ni x\\ L^x \text{ from } \partial_{\rm v}^{\rm ext}S_1 \text{ to } \partial_{\rm v}^{\rm ext}S_2}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\le \\ &\le\; e^{-\frac{m}{2}({\rm dist_h}(S_1,x)+{\rm dist_h}(S_2,x)-1)}\,\bar\gamma_{S_1,S_2,x}\,\cdot \\ &\qquad\cdot \max_{\substack{\mathcal{T}\text{ tree over }\{1,\dots,n\}\\ \mathcal{T}\ni(1,2)}} \prod_{\substack{(i,j)\in\mathcal{T}\\ (i,j)\ne(1,2)}} e^{-\frac{m}{2}({\rm dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}\;. \end{aligned} \tag{3.35}$$

Now using (3.27) and (3.35) we can bound Σ_2'''':

$$\begin{aligned}\Sigma_2'''' := & \sum_{n\geq 2}\frac{1}{n!}\sum_{(S_i)_{i=1}^n}\left(\prod_i e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right)\cdot \\ & \cdot \sum^{*}_{\substack{(L_k)_k,\,\cup_k L_k\ni x \\ L^x \text{ from a } \partial_\mathrm{v}^{\mathrm{ext}} S_i \text{ to a } \partial_\mathrm{v}^{\mathrm{ext}} S_j \text{ with } i\neq j}} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \leq \\ \leq & \sum_{n\geq 2}\sum_{\substack{\mathcal{T}\text{ tree over }\{1,\dots,n\}\\ \mathcal{T}\ni(1,2)}}\frac{n(n-1)}{2\,n!}\sum_{(S_i)_{i=1}^n}\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)|S_i|-\beta J}\right)\cdot \\ & \cdot\, e^{-\frac{m}{2}(\mathrm{dist_h}(S_1,x)+\mathrm{dist_h}(S_2,x)-1)}\,\bar{\gamma}_{S_1,S_2,x}\prod_{\substack{(i,j)\in\mathcal{T}\\(i,j)\neq(1,2)}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}\,.\end{aligned} \tag{3.36}$$

Removing the edge $(1,2)$ from the tree $\mathcal{T}$ one obtains two disjoint trees $\mathcal{T}_1,\mathcal{T}_2$. By applying to each tree the entropy bound (3.11), one finds:

$$\begin{aligned}&\#\big\{(S_i)_{i=1}^n\in\mathscr{PS}_\Lambda\;\big|\;\mathrm{dist_h}(S_1,x)=d_1\,,\ \mathrm{dist_h}(S_2,x)=d_2\,,\ |S_i|=s_i\ \forall i\,, \\ &\qquad\qquad \mathrm{dist_h}(S_i,S_j)=d_{ij}\ \forall(i,j)\in\mathcal{T}\smallsetminus(1,2)\big\} = \\ &= \prod_{t=1,2}\#\big\{(S_i)_{i\in\mathcal{T}_t}\in\mathscr{PS}_\Lambda\;\big|\;\mathrm{dist_h}(S_t,x)=d_t\,,\ |S_i|=s_i\ \forall i\in\mathcal{T}_t\,, \\ &\qquad\qquad \mathrm{dist_h}(S_i,S_j)=d_{ij}\ \forall(i,j)\in\mathcal{T}_t\big\} \leq \\ &\leq \prod_{i=1}^n\left(\frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\ ;\end{aligned}$$

then substitute this entropy bound into (3.36) and obtain:

$$\begin{aligned}\Sigma_2'''' \leq & \sum_{n\geq 2}\sum_{\substack{\mathcal{T}\text{ tree over }\{1,\dots,n\}\\ \mathcal{T}\ni(1,2)}}\frac{n(n-1)}{2\,n!}\sum_{\substack{(s_i)_{i=1,\dots,n}\\ s_i\text{ even }\geq 2}}\ \sum_{d_1,d_2\geq 1}\ \sum_{\substack{(d_{ij})_{(i,j)\in\mathcal{T}\smallsetminus(1,2)}\\ d_{ij}\geq 2}} \\ & \left(\prod_{i=1}^n\frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)s_i-\beta J}\right)\cdot \\ & \cdot\, e^{-\frac{m}{2}(d_1+d_2-1)}\,\gamma_{d_1+d_2}\prod_{\substack{(i,j)\in\mathcal{T}\\(i,j)\neq(1,2)}} e^{-\frac{m}{2}(d_{ij}-1)}\,\gamma_{d_{ij}}\end{aligned} \tag{3.37}$$

where $\gamma_d := \left(\frac{1}{2}e^{-\beta J}+e^{-\beta\frac{\mu_\mathrm{h}+J}{2}\,(d-1)}\right)(1+o(1))\,.$
As already seen, given $n\geq 2$ and $\delta_1,\dots,\delta_n\geq 1$, the number of trees $\mathcal{T}$ over the vertices $\{1,\dots,n\}$ with fixed degrees $\deg_\mathcal{T}(i)=\delta_i\ \ \forall i=1,\dots,n$ is

bounded by $\frac{(n-2)!}{(\delta_1-1)!\cdots(\delta_n-1)!}$. Furthermore the number of edges of $\mathcal{T}$ different from $(1,2)$ is $n-2$. Therefore the bound (3.37) leads to:

$$\begin{aligned}\Sigma_2'''' \le \frac{1}{2}\sum_{n\ge 2}\left(\frac{32}{3}\,e^{-\beta J}\sum_{\substack{s\ge 2\\ \text{even}}} e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-5\log 4\right)s}\sum_{\delta\ge 1}\frac{s^\delta}{(\delta-1)!}\right)^n\cdot\\ \cdot\left(\sum_{d\ge 2} e^{-\frac{m}{2}(d-1)}\,\gamma_d\right)^{n-2}\cdot\sum_{d_1,d_2\ge 1} e^{-\frac{m}{2}(d_1+d_2-1)}\,\gamma_{d_1+d_2}\,.\end{aligned} \tag{3.38}$$

The sums over s,d have been already computed in (3.22), (3.23) respectively; the sum over d_1,d_2 gives, as $\beta\to\infty$,

$$\begin{aligned}\sum_{d_1,d_2\ge 1} e^{-\frac{m}{2}(d_1+d_2-1)}\,\gamma_{d_1+d_2} &= \left(\frac{1}{\left(1-e^{-\frac{m}{2}}\right)^2}\,\frac{e^{-\beta J}}{2}\,+\,o(1)\right)(1+o(1))\\ &= 2\,e^{\beta(\mu_\mathrm{h}+2J)}\,(1+o(1))\,.\end{aligned} \tag{3.39}$$

Substitute (3.22), (3.23), (3.39) into (3.38) and obtain

$$\Sigma_2'''' \le \sum_{n\ge 2}\left(\frac{2^{26}e^2}{3}\,e^{-\beta(\mu_\mathrm{h}-\mu_\mathrm{v})+\beta\frac{\mu_\mathrm{h}+J}{2}}\,(1+o(1))\right)^n e^{\beta J}\,(1+o(1))\,. \tag{3.40}$$

Assume $\boldsymbol{\mu}_\mathrm{h}-\boldsymbol{\mu}_\mathrm{v} > \frac{\boldsymbol{\mu}_\mathrm{h}+\boldsymbol{J}}{\mathbf{2}}$. Then for β sufficiently large the (3.40) becomes:

$$\begin{aligned}\Sigma_2'''' &\le \left(\frac{2^{26}e^2}{3}\,e^{-\beta(\mu_\mathrm{h}-\mu_\mathrm{v})+\beta\frac{\mu_\mathrm{h}+J}{2}}\right)^2 e^{\beta J}\,(1+o(1))\\ &= \frac{2^{52}e^4}{9}\,e^{-\beta\,2(\mu_\mathrm{h}-\mu_\mathrm{v})+\beta(\mu_\mathrm{h}+2J)}\,(1+o(1))\,.\end{aligned} \tag{3.41}$$

In conclusion, by using the estimates (3.17), (3.25), (3.26), (3.30), (3.34), (3.41), and the fact that $m = e^{-\beta\frac{\mu_\mathrm{h}+3J}{2}}\,(1+o(1))$ (see lemma A.5), if we

assume $\boldsymbol{\mu}_{\mathrm{h}} - \boldsymbol{\mu}_{\mathrm{v}} > \frac{\boldsymbol{\mu}_{\mathrm{h}}+\boldsymbol{J}}{\boldsymbol{2}}$, we find that:

$$
\begin{aligned}
&\frac{1}{m} \sum_{\substack{P \in \mathscr{P}_\Lambda \\ \operatorname{supp} P \ni x}} \tilde{\varrho}(P)\, e^{a(P)} \;=\\
&= e^{\beta \frac{\mu_{\mathrm{h}}+3J}{2}} \left(\Sigma_1' + \Sigma_1'' + \Sigma_2' + \Sigma_2'' + \Sigma_2''' + \Sigma_2'''' \right) (1+o(1))\\
&\le \Bigg(\frac{2^{24}}{3}\, e^{-\beta(\mu_{\mathrm{h}}-\mu_{\mathrm{v}}) + \beta \frac{\mu_{\mathrm{h}}+J}{2}} \;+\; \frac{2^{52}\, e^4}{9}\, e^{-\beta\, 2(\mu_{\mathrm{h}}-\mu_{\mathrm{v}}) + \beta \frac{\mu_{\mathrm{h}}+2J}{2}} \;+\; \frac{1}{m}\, e^{-\frac{m}{2} N}\\
&\qquad +\; \frac{2^{25.5}}{3}\, e^{-\beta(\mu_{\mathrm{h}}-\mu_{\mathrm{v}}) + \beta \frac{2\mu_{\mathrm{h}}+5J}{2}} \;+\; \frac{2^{52.5}\, e^4}{9}\, e^{-\beta\, 2(\mu_{\mathrm{h}}-\mu_{\mathrm{v}}) + \beta \frac{3\mu_{\mathrm{h}}+6J}{2}}\\
&\qquad +\; \frac{2^{52}\, e^4}{9}\, e^{-\beta\, 2(\mu_{\mathrm{h}}-\mu_{\mathrm{v}}) + \beta \frac{3\mu_{\mathrm{h}}+7J}{2}} \Bigg) (1+o(1))\\
&= \left(\frac{1}{m}\, e^{-\frac{m}{2} N} \;+\; \frac{2^{25.5}}{3}\, e^{\beta(\mu_{\mathrm{v}} + \frac{5J}{2})} \right) (1+o(1))
\end{aligned}
\tag{3.42}
$$

where N is the minimum distance between two different vertical components of $\partial\Lambda$ and $o(1) \to 0$ as $\beta \to \infty$ (uniformly with respect to N).
Now we assume that $\boldsymbol{\mu}_{\mathrm{v}} + \frac{\boldsymbol{5J}}{\boldsymbol{2}} < \boldsymbol{0}$. Thus there exists $\beta_0 > 0$ such that for all $\beta > \beta_0$ the function $1 + o(1)$ on the r.h.s. of (3.42) is < 2 and the term $\frac{2^{25.5}}{3}\, e^{\beta(\mu_{\mathrm{v}} + \frac{5J}{2})} \le 1/32$. There exists[17] also $N_0(\beta)$ such that for all $N > N_0(\beta)$ the term $\frac{1}{m}\, e^{-\frac{m}{2} N} \le 1/32$. Therefore if $\mu_{\mathrm{v}} + \frac{5J}{2} < 0$ (which entails also the previous condition $\mu_{\mathrm{h}} - \mu_{\mathrm{v}} > \frac{\mu_{\mathrm{h}}+J}{2}$), then the inequality (3.42) implies that

$$
\sum_{\substack{P \in \mathscr{P}_\Lambda \\ \operatorname{supp} P \ni x}} \tilde{\varrho}(P)\, e^{a(P)} \;\le\; \frac{m}{8}
$$

for $\beta > \beta_0$ and $N > N_0(\beta)$. This concludes the proof. □

4 Proofs of the Liquid Crystal Properties

In this section we will finally prove that the model behaves like a liquid crystal, as stated in the section 1, by means of the cluster expansion results obtained in the previous sections.

4.1 Proof of the theorem 1.2

We will prove the inequality (1.10) for $f_{\mathrm{l},x}$. That one for $f_{\mathrm{r},x}$ can be proved analogously; then (1.11) and (1.12) follow since $f_x = f_{\mathrm{l},x} + f_{\mathrm{r},x}$.

[17] $N_0 = \frac{2}{m} \log \frac{32}{m}$.

Observe that

$$\langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus x}}{Z^{\mathrm{h}}_\Lambda}\ ,$$

where $Z^{\mathrm{h}}_{\Lambda\setminus x}$ is the partition function over the lattice $\Lambda\setminus x$ with horizontal boundary conditions including a left-dimer at the site x. Since $N > N_0(\beta)$ and $\mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda) > N_0(\beta)$, both partition functions satisfy the hypothesis of the corollary 3.2. Hence by the cluster expansion (3.4) the partition functions rewrite as

$$Z^{\mathrm{h}}_\Lambda \;=\; C_\Lambda\ \exp\Bigg(\sum_{(P_t)_t\in\mathscr{C}\mathscr{P}_\Lambda}^{\;*} U_\Lambda\big((P_t)_t\big)\Bigg)\ ,$$

$$Z^{\mathrm{h}}_{\Lambda\setminus x} \;=\; C_{\Lambda\setminus x}\ \exp\Bigg(\sum_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x}}^{\;*} U_{\Lambda\setminus x}\big((P_t)_t\big)\Bigg)\ .$$

By applying the definition (2.16),

$$\frac{C_{\Lambda\setminus x}}{C_\Lambda} \;=\; \frac{b_{\mathrm{r},x-(1,0)}\ b_{\mathrm{l},x+(1,0)}}{\lambda_1}\ .$$

Now consider a polymer $P\in P_\Lambda\cup\mathscr{P}_{\Lambda\setminus x}$. Keeping in mind the definitions of polymer (2.12) and polymer activity (2.17), observe that[18]

$$\text{if }\ \mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P,x)>1 \quad\Rightarrow\quad P\in\mathscr{P}_\Lambda\cap\mathscr{P}_{\Lambda\setminus x}\ ,\ \varrho_\Lambda(P)=\varrho_{\Lambda\setminus x}(P)\ .$$

Therefore:

$$\begin{aligned}
&\sum_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x}}^{\;*} U_{\Lambda\setminus x}\big((P_t)_t\big) \;-\; \sum_{(P_t)_t\in\mathscr{C}\mathscr{P}_\Lambda}^{\;*} U_\Lambda\big((P_t)_t\big) \;\geq\\
&\geq\; -\sum_{\substack{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x}\\ \exists t:\,\mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P_t,\,x)\leq 1}}^{\;*} \big|U_{\Lambda\setminus x}\big((P_t)_t\big)\big| \;-\; \sum_{\substack{(P_t)_t\in\mathscr{C}\mathscr{P}_\Lambda\\ \exists t:\,\mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P_t,\,x)\leq 1}}^{\;*} \big|U_\Lambda\big((P_t)_t\big)\big|\ .
\end{aligned}$$

And by the inequalities (3.6) and (3.2) applied to both Z^{h}_Λ, $Z^{\mathrm{h}}_{\Lambda\setminus x}$,

$$\sum_{\substack{(P_t)_t\in\mathscr{C}\mathscr{P}_\Lambda\\ \exists t:\,\mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P_t,\,x)\leq 1}}^{\;*} \big|U_\Lambda\big((P_t)_t\big)\big| \;\leq\; \sum_{\substack{P\in\mathscr{P}_\Lambda\\ \mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P,x)\leq 1}} \widetilde{\varrho}(P)\,e^{a(P)} \;\leq\; 3\,\frac{m}{8}\ ;$$

$$\sum_{\substack{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x}\\ \exists t:\,\mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P_t,\,x)\leq 1}}^{\;*} \big|U_{\Lambda\setminus x}\big((P_t)_t\big)\big| \;\leq\; \sum_{\substack{P\in\mathscr{P}_{\Lambda\setminus x}\\ \mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P,x)\leq 1}} \widetilde{\varrho}(P)\,e^{a(P)} \;\leq\; 2\,\frac{m}{8}\ .$$

[18] The condition $\mathrm{dist}_{\mathrm{h}}(\operatorname{supp}P,x)>1$ guarantees that $\operatorname{supp}P\subseteq\Lambda\setminus x$ and that the polymer P does not include any line L_k having one endpoint on $x\pm(1,0)$, nor any region S_i containing these points.

In conclusion one obtains:

$$\langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}} = \frac{Z^{\mathrm{h}}_{\Lambda\setminus x}}{Z^{\mathrm{h}}_\Lambda} \geq \frac{b_{\mathrm{r},\,x-(1,0)}\, b_{\mathrm{l},\,x+(1,0)}}{\lambda_1} \exp\left(-5\,\frac{m}{8}\right)$$
$$= \frac{1}{2}\left(1 - e^{-\beta\frac{\mu_{\mathrm{h}}+J}{2}}(1+o(1))\right),$$

where the last identity follows from the fact that $\lambda_1\, b_{\mathrm{r},x-(1,0)}\, b_{\mathrm{l},x+(1,0)} = E_{\mathrm{l}}^{(1)} B_{\mathrm{r},x-(1,0)}\, B_{\mathrm{l},x+(1,0)} E_{\mathrm{r}}^{(1)} = \frac{1}{\sqrt{2}}(1 - \frac{a}{2}\,(1+o(1)))\,\frac{1}{\sqrt{2}}(1 - \frac{a}{2}\,(1+o(1)))$ (by lemma A.8, since there is a left-dimer fixed at x according to $Z^{\mathrm{h}}_{\Lambda\setminus x}$), $\lambda_1 = 1 + \frac{ab}{2}\,(1+o(1))$ (proposition A.3), and $e^{-5m/8} = 1 - \frac{5}{8}ab\,(1+o(1))$ (lemma A.5). Finally, since $o(1)\to 0$ as $\beta\to\infty$ and $o(1)$ does not depend on the choice of x and Λ, one may obtain the desired inequality eventually increasing β_0. □

4.2 Proof of the corollary 1.4

Set $\varphi_{\Lambda,N_0} := \#\{x \in \Lambda \,|\, \mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda) > N_0\}\,/\,|\Lambda|$. By the theorem 1.2, bound (1.11), using also $f_{\mathrm{v},x} \leq 1 - f_{\mathrm{h},x}$, one obtains:

$$\langle \Delta_{\mathrm{orient.}}\rangle_\Lambda^{\mathrm{h}} = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left(\langle f_{\mathrm{h},x}\rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{v},x}\rangle_\Lambda^{\mathrm{h}}\right) \geq \varphi_{\Lambda,N_0(\beta)}\left(1 - 4\,e^{-\beta\frac{\mu_{\mathrm{h}}+J}{2}}\right).$$

On the other hand:

$$\varphi_{\Lambda,N_0} \geq \min_{\substack{L \text{ maximal} \\ \text{horiz. line of } \Lambda}} \varphi_{L,N_0} = \min_{\substack{L \text{ maximal} \\ \text{horiz. line of } \Lambda}} \frac{|L| - 2N_0(\beta)}{|L|} = 1 - 2\,\frac{N_0(\beta)}{N}.$$

□

4.3 Proof of the corollary 1.5

Set $\varphi_{\Lambda,N_0} := \#\{x \in \Lambda \,|\, \mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda) > N_0\}\,/\,|\Lambda|$. By the theorem 1.2, bound (1.12),

$$\left|\langle \Delta_{\mathrm{transl.}}\rangle_\Lambda^{\mathrm{h}}\right| \leq \frac{2}{|\Lambda|}\sum_{\substack{x\in\Lambda,\\ x_{\mathrm{h}}\text{ even}}}\left|\langle f_{\mathrm{r},x}\rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}}\right| \leq \varphi_{\Lambda,N_0(\beta)}\, 2e^{-\beta\frac{\mu_{\mathrm{h}}+J}{2}} + 1 - \varphi_{\Lambda,N_0(\beta)}.$$

On the other hand we have already observed in the proof of the corollary 1.4 that $\varphi_{\Lambda,N_0} \geq 1 - 2N_0/N$. □

4.4 Proof of the theorem 1.3

We will prove the inequality (1.13). (1.14) and (1.15) can be proved analogously. First of all observe that, since $0 \le f_{1,x}, f_{1,y} \le 1$,

$$\big|\langle f_{1,x}\, f_{1,y}\rangle^{\mathrm{h}}_{\Lambda} - \langle f_{1,x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}\big| \;\le\; \log\left(\frac{\langle f_{1,x}\, f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}}{\langle f_{1,x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}} \;\vee\; \frac{\langle f_{1,x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}}{\langle f_{1,x}\, f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}} \right) . \tag{4.1}$$

Now observe that:

$$\langle f_{1,x}\, f_{1,y}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus x,y}}{Z^{\mathrm{h}}_{\Lambda}} \;, \quad \langle f_{1,x}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus x}}{Z^{\mathrm{h}}_{\Lambda}} \;, \quad \langle f_{1,y}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus y}}{Z^{\mathrm{h}}_{\Lambda}} \;,$$

where $Z^{\mathrm{h}}_{\Lambda\setminus x}$, $Z^{\mathrm{h}}_{\Lambda\setminus y}$, $Z^{\mathrm{h}}_{\Lambda\setminus x,y}$ are the partition function respectively over the lattices $\Lambda\setminus x$, $\Lambda\setminus y$, $\Lambda\setminus x, y$, with horizontal boundary conditions including a left-dimer respectively at the site x , at the site y , at both sites x, y . Therefore

$$\frac{\langle f_{1,x}\, f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}}{\langle f_{1,x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{1,y}\rangle^{\mathrm{h}}_{\Lambda}} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda}\; Z^{\mathrm{h}}_{\Lambda\setminus x,y}}{Z^{\mathrm{h}}_{\Lambda\setminus x}\; Z^{\mathrm{h}}_{\Lambda\setminus y}} \;. \tag{4.2}$$

Since $N > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda) > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(y,\partial\Lambda) > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(x,y) > N_0(\beta)$, all four partition functions satisfy the hypothesis of the corollary 3.2. Hence by the cluster expansion (3.4) the partition functions rewrites as

$$\begin{aligned}
Z^{\mathrm{h}}_{\Lambda} \;&=\; C_{\Lambda}\, \exp\bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda}} U_{\Lambda}\big((P_t)_t\big) \bigg) \;, \\
Z^{\mathrm{h}}_{\Lambda\setminus x} \;&=\; C_{\Lambda\setminus x}\, \exp\bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x}} U_{\Lambda\setminus x}\big((P_t)_t\big) \bigg) \;, \\
Z^{\mathrm{h}}_{\Lambda\setminus y} \;&=\; C_{\Lambda\setminus y}\, \exp\bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus y}} U_{\Lambda\setminus y}\big((P_t)_t\big) \bigg) \;, \\
Z^{\mathrm{h}}_{\Lambda\setminus x,y} \;&=\; C_{\Lambda\setminus x,y}\, \exp\bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x,y}} U_{\Lambda\setminus x,y}\big((P_t)_t\big) \bigg) \;.
\end{aligned} \tag{4.3}$$

By applying the definition (2.16), it holds

$$\frac{C_{\Lambda}\; C_{\Lambda\setminus x,y}}{C_{\Lambda\setminus x}\; C_{\Lambda\setminus y}} \;=\; 1 \;. \tag{4.4}$$

Now consider a polymer $P \in \mathscr{P}_{\Lambda} \cup \mathscr{P}_{\Lambda\setminus x} \cup \mathscr{P}_{\Lambda\setminus y} \cup \mathscr{P}_{\Lambda\setminus x,y}$. Keeping in mind the definitions of polymer (2.12) and polymer activity (2.17), observe that:

if $\mathrm{dist}_{\mathrm{h}}(\mathrm{supp}\, P, x) > 1$, $\mathrm{dist}_{\mathrm{h}}(\mathrm{supp}\, P, y) > 1 \quad\Rightarrow$
$P \in \mathscr{P}_{\Lambda} \cap \mathscr{P}_{\Lambda\setminus x} \cap \mathscr{P}_{\Lambda\setminus y} \cap \mathscr{P}_{\Lambda\setminus x,y}$, $\varrho_{\Lambda}(P) = \varrho_{\Lambda\setminus x}(P) = \varrho_{\Lambda\setminus y}(P) = \varrho_{\Lambda\setminus x,y}(P)$;

and that[19]:

if $\mathrm{dist_h}(\mathrm{supp}\, P, x) \leq 1$, $\mathrm{dist_h}(\mathrm{supp}\, P, y) > 1 \quad \Rightarrow$
$P \in \big(\mathscr{P}_\Lambda \cap \mathscr{P}_{\Lambda\setminus y}\big) \setminus \big(\mathscr{P}_{\Lambda\setminus x} \cup \mathscr{P}_{\Lambda\setminus x,y}\big)$, $\varrho_\Lambda(P) = \varrho_{\Lambda\setminus y}(P)$ or
$P \in \big(\mathscr{P}_{\Lambda\setminus x} \cap \mathscr{P}_{\Lambda\setminus x,y}\big) \setminus \big(\mathscr{P}_\Lambda \cup \mathscr{P}_{\Lambda\setminus y}\big)$, $\varrho_{\Lambda\setminus x}(P) = \varrho_{\Lambda\setminus x,y}(P)$ or
$P \in \mathscr{P}_\Lambda \cap \mathscr{P}_{\Lambda\setminus x} \cap \mathscr{P}_{\Lambda\setminus y} \cap \mathscr{P}_{\Lambda\setminus x,y}$, $\varrho_\Lambda(P) = \varrho_{\Lambda\setminus y}(P)$, $\varrho_{\Lambda\setminus x}(P) = \varrho_{\Lambda\setminus x,y}(P)$;

and the case $\mathrm{dist_h}(\mathrm{supp}\, P, x) > 1$, $\mathrm{dist_h}(\mathrm{supp}\, P, y) \leq 1$ is clearly symmetric to the previous one. Therefore:

$$\begin{aligned}
&\sideset{}{^*}\sum_{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda} U_\Lambda\big((P_t)_t\big) \;-\; \sideset{}{^*}\sum_{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x}} U_{\Lambda\setminus x}\big((P_t)_t\big) \;+\\
&-\sideset{}{^*}\sum_{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus y}} U_{\Lambda\setminus y}\big((P_t)_t\big) \;+\; \sideset{}{^*}\sum_{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x,y}} U_{\Lambda\setminus x,y}\big((P_t)_t\big) \;\leq\\
&\leq \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \leq 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \leq 1}} \big|U_\Lambda\big((P_t)_t\big)\big| \;+\; \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x} \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \leq 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \leq 1}} \big|U_{\Lambda\setminus x}\big((P_t)_t\big)\big| \;+ \qquad (4.5)\\
&+ \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus y} \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \leq 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \leq 1}} \big|U_{\Lambda\setminus y}\big((P_t)_t\big)\big| \;+\; \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x,y} \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \leq 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \leq 1}} \big|U_{\Lambda\setminus x,y}\big((P_t)_t\big)\big| \;.
\end{aligned}$$

It is crucial to observe that given a cluster $(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda$, since $\cup_t \mathrm{supp}\, P_t$ have to be connected in $\mathbb{Z}^2$,

$$\mathrm{dist}_{\mathbb{Z}^2}(x,y) \;\leq\; \mathrm{dist}_{\mathbb{Z}^2}(\cup_t \mathrm{supp}\, P_t, x) + \sum_t |\,\mathrm{supp}\, P_t| - 1 + \mathrm{dist}_{\mathbb{Z}^2}(\cup_t \mathrm{supp}\, P_t, y) \;.$$

[19] The first possibility, namely P polymer only of the lattices that contain x, happens when $\mathrm{supp}\, P \ni x$ or P includes a region S_i containing $x - (1,0)$. The second possibility, namely P polymer only of the lattices that do not contain x, happens when P includes a line L_k with one endpoint on $x \pm (1,0)$. The last possibility happens when P includes a region S_i containing $x + (1,0)$ (and does not verify the other conditions).

Hence, assuming that $\mathrm{dist}_{\mathbb{Z}^2}(\cup_t \operatorname{supp} P_t, x) \le 1$, $\mathrm{dist}_{\mathbb{Z}^2}(\cup_t \operatorname{supp} P_t, y) \le 1$, it follows

$$
\begin{aligned}
&\prod_t \widetilde{\varrho}(P_t) \; = \\
&= \prod_t \frac{1}{n_t!\, p_t!} \, \exp\left(-\beta \frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} \sum_{i=1}^{n_t} |S_i| - m \sum_{k=1}^{p_t} |L_k| - \beta J \, n_t \right) \\
&= \exp\left(-\frac{m}{4} \sum_t |\operatorname{supp} P_t| \right) \cdot \\
&\quad \cdot \prod_t \frac{1}{n_t!\, p_t!} \, \exp\left(-\left(\beta \frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} - \frac{m}{4} \right) \sum_{i=1}^{n_t} |S_i| - \frac{3\,m}{4} \sum_{k=1}^{p_t} |L_k| - \beta J \, n_t \right) \\
&\le \exp\Big(-\frac{m}{4} \left(\mathrm{dist}_{\mathbb{Z}^2}(x,y) - 1 \right) \Big) \prod_t \widetilde{\varrho}_*(P_t)
\end{aligned}
$$

where $P_t = \big((S_i)_{i=1}^{n_t}, (L_k)_{k=1}^{p_t} \big)$ for all t and $\widetilde{\varrho}_*(P_t)$ is defined as the factor appearing in the product over t at the penultimate step. By defining $a_*(P) := \frac{m}{4} |\operatorname{supp} P|$, we have that $\widetilde{\varrho}_*(P)\, e^{a_*(P)}$ is essentially equivalent to $\widetilde{\varrho}(P)\, e^{a(P)}$: we can follow exactly the proof of the theorem 3.1 up to the inequality (3.42) and prove that the Kotecky-Preiss conditions (3.2), (3.3) hold also with $\widetilde{\varrho}_*$, a_* and $m/16$ in place of $\widetilde{\varrho}$, a and $m/8$ (eventually increasing β_0). Therefore, defining $\widetilde{U}_*\big((P_t)_t\big) := u\big((P_t)_t\big) \prod_t \widetilde{\varrho}_*(P_t)$, by the general theory of cluster expansion the inequality (3.6) holds also with $\widetilde{U}_*$, $\widetilde{\varrho}_*$ and a_* in place of U_Λ, ϱ_Λ and a . As a consequence:

$$
\begin{aligned}
&\sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t\colon \mathrm{dist_h}(\operatorname{supp} P_t, x) \le 1 \\ \exists t'\colon \mathrm{dist_h}(\operatorname{supp} P_{t'}, y) \le 1}} \big| U_\Lambda\big((P_t)_t\big) \big| \;\le\; \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t\colon \mathrm{dist_h}(\operatorname{supp} P_t, x) \le 1 \\ \exists t'\colon \mathrm{dist_h}(\operatorname{supp} P_{t'}, y) \le 1}} \big| u\big((P_t)_t\big) \big| \prod_t \widetilde{\varrho}(P_t) \;\le \\
&\le\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t\colon \mathrm{dist_h}(\operatorname{supp} P_t, x) \le 1 \\ \exists t'\colon \mathrm{dist_h}(\operatorname{supp} P_{t'}, y) \le 1}} \big| u\big((P_t)_t\big) \big| \prod_t \widetilde{\varrho}_*(P_t) \\
&=\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t\colon \mathrm{dist_h}(\operatorname{supp} P_t, x) \le 1 \\ \exists t'\colon \mathrm{dist_h}(\operatorname{supp} P_{t'}, y) \le 1}} \big| \widetilde{U}_*\big((P_t)_t\big) \big| \qquad\qquad (4.6) \\
&\overset{(3.6)}{\le}\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \sum_{\substack{P \in \mathscr{P}_\Lambda \\ \mathrm{dist_h}(\operatorname{supp} P, x) \le 1}} \widetilde{\varrho}_*(P)\, e^{a_*(P)} \\
&\overset{(3.2)}{\le}\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \; 3\, \frac{m}{16} \; .
\end{aligned}
$$

The same reasoning can be repeated also for the clusters in $\mathscr{CP}_{\Lambda\setminus x}$, $\mathscr{CP}_{\Lambda\setminus y}$ and $\mathscr{CP}_{\Lambda\setminus x,y}$. Thus, by (4.2), (4.3), (4.4), 4.5, (4.6), one finally obtains:

$$\frac{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}}{\langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}} \;=\; \frac{Z_\Lambda^{\mathrm{h}}\; Z_{\Lambda\setminus x,y}^{\mathrm{h}}}{Z_{\Lambda\setminus x}^{\mathrm{h}}\; Z_{\Lambda\setminus y}^{\mathrm{h}}} \;\leq\; \exp\left(e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)}\;(3+2+2+2)\,\frac{m}{16}\right)\;.$$

The same bound can be shown to hold also for the inverse ratio $\frac{\langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}}{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}}$, hence by (4.1) we conclude that:

$$\left|\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}} \;-\; \langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}\right| \;\leq\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)}\;\frac{9m}{16}\;.$$

$\square$

4.5 Proof of the corollary 1.6

Since $\Delta_{\mathrm{transl.}} = \frac{2}{|\Lambda|}\sum_{\substack{x\in\Lambda,\\ x_{\mathrm{h}}\ \mathrm{even}}}(f_{\mathrm{r},x} - f_{\mathrm{l},x})$, the variance of Δ rewrites as:

$$\left\langle(\Delta_{\mathrm{transl.}})^2\right\rangle_\Lambda^{\mathrm{h}} \;-\; \left(\langle\Delta_{\mathrm{transl.}}\rangle_\Lambda^{\mathrm{h}}\right)^2 \;=\; \frac{4}{|\Lambda|^2}\sum_{\substack{x,y\in\Lambda\\ x_{\mathrm{h}},y_{\mathrm{h}}\ \mathrm{even}}} C_{x,y}$$

with

$$\begin{aligned} C_{x,y} \;:=\; &\left(\langle f_{\mathrm{r},x}\, f_{\mathrm{r},y}\rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{r},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{r},y}\rangle_\Lambda^{\mathrm{h}}\right) + \left(\langle f_{\mathrm{r},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{r},x}\, f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}\right) + \\ &+ \left(\langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{r},y}\rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{l},x}\, f_{\mathrm{r},y}\rangle_\Lambda^{\mathrm{h}}\right) + \left(\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{l},x}\rangle_\Lambda^{\mathrm{h}}\,\langle f_{\mathrm{l},y}\rangle_\Lambda^{\mathrm{h}}\right)\;. \end{aligned}$$

By the theorem 1.3, for $x,y\in\Lambda$ such that $\mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda) > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(y,\partial\Lambda) > N_0(\beta)$ and $\mathrm{dist}_{\mathrm{h}}(x,y) > N_0(\beta)$, it holds

$$C_{x,y} \;\leq\; 4\,\frac{9m}{16}\;e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)}\;.$$

Hence:

$$\left\langle(\Delta_{\mathrm{transl.}})^2\right\rangle_\Lambda^{\mathrm{h}} - \left(\langle\Delta_{\mathrm{transl.}}\rangle_\Lambda^{\mathrm{h}}\right)^2 \;\leq\; 4\,\frac{9m}{16|\Lambda|^2}\sum_{\substack{x,y\in\Lambda\\ x\neq y}} e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} + 1 - \varphi_{\Lambda,\Lambda,N_0(\beta)}\;,$$

where we set

$$\varphi_{\Lambda,\Lambda',N_0} := \frac{\#\{(x,y)\in\Lambda\times\Lambda' \mid \mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda)\vee\mathrm{dist}_{\mathrm{h}}(y,\partial\Lambda')\vee\mathrm{dist}_{\mathrm{h}}(x,y) > N_0\}}{|\Lambda|\;|\Lambda'|}\;.$$

Now observe that

$$\begin{aligned} \varphi_{\Lambda,\Lambda,N_0} \;&\geq\; \min_{\substack{L,L'\ \mathrm{maximal}\\ \mathrm{horiz.\,lines\ of}\ \Lambda}} \varphi_{L,L',N_0} \;\geq\; \min_{\substack{L,L'\ \mathrm{maximal}\\ \mathrm{horiz.\,lines\ of}\ \Lambda}} \frac{(|L|-2N_0)\,(|L'|-4N_0)}{|L|\,|L'|} \\ &\geq\; \left(1 - 2\frac{N_0}{N}\right)\left(1 - 4\frac{N_0}{N}\right)\;, \end{aligned}$$

hence $1-\varphi_{\Lambda,\Lambda,N_0} \le N_0/N\,(6-8N_0/N)$. And on the other hand:

$$\sum_{\substack{x,y\in\Lambda\\ x\neq y}} e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \;\le\; |\Lambda| \sum_{\substack{x\in\mathbb{Z}^2\\ x\neq 0}} e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,0)-1)} \;=$$

$$=\;|\Lambda| \sum_{d\ge 1} 4d\, e^{-\frac{m}{4}(d-1)} \;=\; |\Lambda|\,\frac{4}{(1-e^{-\frac{m}{4}})^2}\;.$$

□

A Appendix: 1D Systems

Consider a finite line L, that is a finite connected sub-lattice of $\mathbb{Z}$. Consider a monomer-dimer model on L given by the following partition function:

$$Z_L \;=\; \sum_{\alpha\in\mathscr{D}_L} e^{-\beta H_L(\alpha)}\; e^{I_\mathrm{l}(\alpha_{x_\mathrm{l}})}\; e^{I_\mathrm{r}(\alpha_{x_\mathrm{r}})}\;.$$

$\mathscr{D}_L$ denotes the set of monomer-dimer configurations on L (allowing also external dimers at the endpoints of L); the Hamiltonian is defined as

$$H_L \;=\; \frac{\mu_\mathrm{h}+J}{2}\;\#\begin{Bmatrix}\text{sites of } L \text{ with}\\ \text{monomer}\end{Bmatrix} \;+\; \frac{J}{2}\;\#\begin{Bmatrix}\text{sites of } L \text{ with dimer but}\\ \text{neighbor to monomer in } L\end{Bmatrix}\;.$$

$x_\mathrm{l}, x_\mathrm{r}$ denote the left and the right endpoint of L respectively; $I_\mathrm{l}, I_\mathrm{r}$ represent the interaction among the configuration on L and the boundary condition outside its endpoints.

This one-dimensional system can be described by a *transfer matrix* T over the three possible states of a site, $l\equiv$“left-dimer”, $r\equiv$“right-dimer”, $m\equiv$“monomer”:

$$T \;\equiv\; \begin{pmatrix} T(l,l) & T(l,r) & T(l,m)\\ T(r,l) & T(r,r) & T(r,m)\\ T(m,l) & T(m,r) & T(m,m)\end{pmatrix} \;:=\; \begin{pmatrix} 0 & 1 & \sqrt{ab}\\ 1 & 0 & 0\\ 0 & \sqrt{ab} & a\end{pmatrix}\;, \tag{A.1}$$

where to shorten the notation we set $\sqrt{a} \;:=\; e^{-\beta\frac{\mu_\mathrm{h}+J}{4}}$ the transfer contribution of a monomer[20], $\sqrt{b} \;:=\; e^{-\beta\frac{J}{2}}$ the transfer contribution of a site with a dimer but neighbor to a monomer. Two vectors are also needed to encode the boundary conditions:

$$B_\mathrm{l} \;\equiv\; (B_\mathrm{l}(l)\; B_\mathrm{l}(r)\; B_\mathrm{l}(m)) \;:=\; \begin{pmatrix} e^{I_\mathrm{l}(l)} & e^{I_\mathrm{l}(r)} & \sqrt{a}\, e^{I_\mathrm{l}(m)}\end{pmatrix}\;,$$

$$B_\mathrm{r} \;\equiv\; \begin{pmatrix} B_\mathrm{r}(l)\\ B_\mathrm{r}(r)\\ B_\mathrm{r}(m)\end{pmatrix} \;:=\; \begin{pmatrix} e^{I_\mathrm{r}(l)}\\ e^{I_\mathrm{r}(r)}\\ \sqrt{a}\, e^{I_\mathrm{r}(m)}\end{pmatrix}\;. \tag{A.2}$$

Proposition A.1 *The partition function of the system rewrites as a bilinear form:*

$$Z_L \;=\; B_\mathrm{l}\, T^{|L|-1}\, B_\mathrm{r}\;. \tag{A.3}$$

[20] The transfer energy of a monomer is half the energy of a monomer because it appears during two “transfers”.

Proof According to the previous definitions it is clear that for every configuration $\alpha \in \{l, r, m\}^{|L|}$

$$\begin{aligned}
&\mathbb{1}(\alpha \in \mathscr{D}_L)\; e^{-\beta H_L(\alpha)} \;= \\
&\qquad = \sqrt{a}^{\,\mathbb{1}(\alpha_1 = m)}\; T(\alpha_1, \alpha_2)\; T(\alpha_2, \alpha_3)\; \ldots\; T(\alpha_{|L|-1}, \alpha_{|L|})\; \sqrt{a}^{\,\mathbb{1}(\alpha_{|L|} = m)} \quad .
\end{aligned}$$

Therefore

$$\begin{aligned}
Z_L \;&=\; \sum_{\alpha \in \{l,r,m\}^{|L|}} B_{\mathrm{l}}(\alpha_1)\; T(\alpha_1, \alpha_2)\; T(\alpha_2, \alpha_3)\; \ldots\; T(\alpha_{|L|-1}, \alpha_{|L|})\; B_{\mathrm{r}}(\alpha_{|L|}) \\
&=\; B_{\mathrm{l}}\, T^{|L|-1}\, B_{\mathrm{r}} \;.
\end{aligned}$$

□

Assume for the moment that the transfer matrix T is diagonalizable. Denote by $\lambda_1, \lambda_2, \lambda_3$ its eigenvalues and by $E_{\mathrm{r}}^{(1)}, E_{\mathrm{r}}^{(2)}, E_{\mathrm{r}}^{(3)}$, $E_{\mathrm{l}}^{(1)}, E_{\mathrm{l}}^{(2)}, E_{\mathrm{l}}^{(3)}$ the corresponding right (column) eigenvectors and left (row) eigenvectors, normalized so that $E_{\mathrm{l}}^{(i)} E_{\mathrm{r}}^{(i)} = 1$ for $i = 1, 2, 3$.

Corollary A.2

$$Z_L \;=\; \sum_{i=1,2,3} \lambda_i^{|L|-1}\, B_{\mathrm{l}}\, E_{\mathrm{r}}^{(i)}\, E_{\mathrm{l}}^{(i)}\, B_{\mathrm{r}} \;. \tag{A.4}$$

Proof Since we are assuming that T is diagonalizable, it holds $T = P\,D\,P^{-1}$ where D is the diagonal matrix of eigenvalues, P is the matrix with the right eigenvectors on the columns, P^{-1} has the left eigenvectors on the rows. Then $T^{|L|-1} = P\,D^{|L|-1}\,P^{-1}$ and

$$B_{\mathrm{l}}\, T^{|L|-1}\, B_{\mathrm{r}} \;=\; (B_{\mathrm{l}}\, P)\, D^{|L|-1}\, (P^{-1}\, B_{\mathrm{r}}) \;=\; \sum_{i=1}^{3} (B_{\mathrm{l}}\, E_{\mathrm{r}}^{(i)})\, \lambda_i^{|L|-1}\, (E_{\mathrm{l}}^{(i)} B_{\mathrm{r}}) \;.$$

□

Now our purpose is to diagonalise the transfer matrix T when β is large.

Proposition A.3 *For all $\beta > 0$ the transfer matrix T is diagonalizable over $\mathbb{R}$. Its eigenvalues are*

$$\begin{aligned}
\lambda_1 \;&=\; 1 + \frac{ab}{2}\,(1 + o(1)) \\
\lambda_2 \;&=\; -1 + \frac{ab}{2}\,(1 + o(1)) \\
\lambda_3 \;&=\; a - ab - a^3 b\,(1 + o(1))
\end{aligned} \tag{A.5}$$

as $\beta \to \infty$.

Proof The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are the (complex) roots of the characteristic polynomial of T, that is

$$p(\lambda) := \det(\lambda I - T) = -ab + (\lambda - a)(\lambda^2 - 1) \;.$$

For all $\beta > 0$ it turns out that p has 3 distinct real roots[21], hence T is diagonalizable over the reals.

[21] The discriminant of the cubic is $\Delta = 18a(1-b) + 4a^2(1-b) + a^2 + 4 - 27a^2(1-b)$, which is strictly positive for all $0 \le a, b \le 1$, $(a, b) \neq (1, 0)$.

As $\beta \to \infty$, $p(\lambda) \to \lambda(\lambda^2 - 1)$ hence $\lambda_1 \to 1$, $\lambda_2 \to -1$, $\lambda_3 \to 0$. Thus it is convenient to write $\lambda_1 = 1 + \varepsilon_1$, $\lambda_2 = -1 + \varepsilon_2$, $\lambda_3 = a + \varepsilon_3$ with $\varepsilon_i \to 0$ as $\beta \to \infty$ for $i = 1, 2, 3$. Now expand the polynomial p in powers of ε_i and truncate it at the first order:

$$
\begin{aligned}
0 \,=\, p(\lambda_1) \,&=\, -ab + (1 - a + \varepsilon_1)\,(2\varepsilon_1 + \varepsilon_1^2) \,=\, -ab + 2\varepsilon_1\,(1 + o(1)) \\
&\Rightarrow \varepsilon_1 = \frac{ab}{2}\,(1 + o(1))\;; \\
0 \,=\, p(\lambda_2) \,&=\, -ab + (-1 - a + \varepsilon)\,(-2\varepsilon_2 + \varepsilon_2^2) \,=\, -ab + 2\varepsilon_2\,(1 + o(1)) \\
&\Rightarrow \varepsilon_2 = \frac{ab}{2}\,(1 + o(1))\;; \\
0 \,=\, p(\lambda_3) \,&=\, -ab + \varepsilon_3\left((a + \varepsilon_3)^2 - 1\right) \,=\, -ab - \varepsilon_3\,(1 + o(1)) \\
&\Rightarrow \varepsilon_3 = -ab\,(1 + o(1))\;.
\end{aligned}
$$

In order to find the following order of λ_3, now one can write $\lambda_3 = a - ab\,(1 + \varepsilon_3')$ with $\varepsilon_3' \to 0$ as $\beta \to \infty$ and repeat the procedure:

$$
\begin{aligned}
0 \,=\, \frac{p(\lambda_3)}{-ab} \,&=\, 1 + (1 + \varepsilon_3')\left(a^2\,(1 + o(1)) - 1\right) \,=\, a^2\,(1 + o(1)) - \varepsilon_3'\,(1 + o(1)) \\
\Rightarrow \varepsilon_3' &= a^2\,(1 + o(1))\;.
\end{aligned}
$$

□

Proposition A.4 *The right eigenvectors of the transfer matrix T are*

$$
\begin{aligned}
E_{\mathrm{r}}^{(1)} \,&=\, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - \frac{a}{2}\,(1 + o(1)) \\ 1 - \frac{a}{2}\,(1 + o(1)) \\ \sqrt{ab}\,(1 + o(1)) \end{pmatrix} \\
E_{\mathrm{r}}^{(2)} \,&=\, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \frac{a}{2}\,(1 + o(1)) \\ -1 - \frac{a}{2}\,(1 + o(1)) \\ \sqrt{ab}\,(1 + o(1)) \end{pmatrix} \\
E_{\mathrm{r}}^{(3)} \,&=\, \begin{pmatrix} -a\sqrt{ab}\,(1 + o(1)) \\ -\sqrt{ab}\,(1 + o(1)) \\ 1 + a\,(1 + o(1)) \end{pmatrix}
\end{aligned}
\tag{A.6}
$$

and moreover

$$
\begin{aligned}
E_{\mathrm{r}}^{(2)}(1) + E_{\mathrm{r}}^{(2)}(2) + \sqrt{ab}\,E_{\mathrm{r}}^{(2)}(3) \,&=\, \frac{ab}{2\sqrt{2}}\,(1 + o(1)) \\
E_{\mathrm{r}}^{(3)}(2) + \sqrt{ab}\,E_{\mathrm{r}}^{(3)}(3) \,&=\, -a^2\sqrt{ab}\,(1 + o(1))
\end{aligned}
$$

as $\beta \to \infty$. The left eigenvectors are obtained by a simple transformation: $E_{\mathrm{l}}^{(i)} = \sigma\big(E_{\mathrm{r}}^{(i)}\big)$ for $i = 1, 2, 3$, where

$$
\sigma \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \,:=\, (v_2\ v_1\ v_3)\;.
$$

Proof The right eigenvectors E_{r} associated to the eigenvalue λ are the non-zero solutions of the linear system

$$
(\lambda I - T)\,E_{\mathrm{r}} = 0 \;\Leftrightarrow\; E_{\mathrm{r}} = \begin{pmatrix} \lambda(\lambda - a) \\ \lambda - a \\ \sqrt{ab} \end{pmatrix} t\;,\; t \in \mathbb{R}\;.
$$

And the left eigenvectors E_l associated to the same eigenvalue λ are the non-zero solutions of the linear system

$$E_\mathrm{l}\,(\lambda I - T) = 0 \;\Leftrightarrow\; E_\mathrm{l} = \big(\lambda - a \;\; \lambda(\lambda - a) \;\; \sqrt{ab}\,\big)\, t \;,\;\; t \in \mathbb{R}\;.$$

The desired normalization $E_\mathrm{l}\,E_\mathrm{r} = 1$ can be obtained by choosing

$$t = \sqrt{2\lambda(\lambda - a) + ab}$$

in both cases. Now to conclude the proof it is sufficient to exploit the estimates of the eigenvalues given by the proposition A.3. □

The formula (A.4) together with the estimates of propositions A.3, A.4 give us a complete control on the one-dimensional system on L at low temperature, for every choice of the boundary conditions.

We concentrate on providing an estimation of the quantity R_L defined by (2.9), since it is needed in the section 2. We have to distinguish three cases, according to where the endpoints of L lie.

Lemma A.5 *The ratios of the eigenvalues of the transfer matrix T are*

$$\frac{\lambda_2}{\lambda_1} \;=\; -1 + ab\,(1 + o(1)) \quad , \quad \frac{\lambda_3}{\lambda_2} \;=\; -a + ab\,(1 + o(1))$$

as $\beta \to \infty$. In particular setting $m := -\log\big|\lambda_2/\lambda_1\big|$ it holds

$$e^{-m} \;=\; 1 - e^{-\beta\frac{\mu_\mathrm{h}+3J}{2}}\,(1 + o(1)) \quad \text{as } \beta \to \infty\;. \tag{A.7}$$

Proof It follows immediately from the proposition A.3. □

Lemma A.6 *If $x_\mathrm{l} \in \partial_\mathrm{r}^\mathrm{ext} S_j$, then as $\beta \to \infty$*

$$B_\mathrm{l}\,E_\mathrm{r}^{(1)} = \frac{\sqrt{b}}{\sqrt{2}}\,(1 + o(1))$$

$$B_\mathrm{l}\,E_\mathrm{r}^{(2)} = -\frac{\sqrt{b}}{\sqrt{2}}\,(1 + o(1))$$

$$B_\mathrm{l}\,E_\mathrm{r}^{(3)} = \sqrt{a}\,(1 + o(1))\;.$$

If $x_\mathrm{r} \in \partial_\mathrm{l}^\mathrm{ext} S_j$, then the same estimates hold for $E_\mathrm{l}^{(1)} B_\mathrm{r}$, $E_\mathrm{l}^{(2)} B_\mathrm{r}$, $E_\mathrm{l}^{(3)} B_\mathrm{r}$ respectively.

Proof If $x_\mathrm{l} \in \partial_\mathrm{r}^\mathrm{ext} S_j$ then by (2.7) and (A.2) the vector describing the boundary condition on the left side of the line L is $B_\mathrm{l} = \big(0 \;\; \sqrt{b} \;\; \sqrt{a}\big)$. Then the estimates for $B_\mathrm{l}\,E_\mathrm{r}^{(i)}$, $i = 1, 2, 3$, are computed using the proposition A.4. □

Lemma A.7 *If $x_\mathrm{l} \in \partial_\mathrm{l}\Lambda$, then as $\beta \to \infty$*

$$B_\mathrm{l}\,E_\mathrm{r}^{(1)} = \begin{cases} \frac{1}{\sqrt{2}}\big(1 - \frac{a}{2}\,(1 + o(1))\big) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ has fixed position} \\ \sqrt{2}\,\big(1 - \frac{a}{2}\,(1 + o(1))\big) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ has free position} \end{cases}$$

$$B_\mathrm{l}\,E_\mathrm{r}^{(2)} = \begin{cases} -\frac{1}{\sqrt{2}}\big(1 + \frac{a}{2}\,(1 + o(1))\big) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ is fixed to the left} \\ \frac{1}{\sqrt{2}}\big(1 + \frac{a}{2}\,(1 + o(1))\big) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ is fixed to the right} \\ \frac{ab}{2\sqrt{2}}\,(1 + o(1)) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ has free position} \end{cases}$$

$$B_\mathrm{l}\,E_\mathrm{r}^{(3)} = \begin{cases} -a^2\sqrt{ab}\,(1 + o(1)) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ is fixed to the left} \\ -a\sqrt{ab}\,(1 + o(1)) & \textit{if the h-dimer on } x_\mathrm{l} - (1,0) \textit{ is fixed to the right or free} \end{cases}$$

If $x_\mathrm{r} \in \partial_\mathrm{r}\Lambda$, then the same estimates hold respectively for $E_\mathrm{l}^{(1)} B_\mathrm{r}$, $E_\mathrm{l}^{(2)} B_\mathrm{r}$, $E_\mathrm{l}^{(3)} B_\mathrm{r}$ after substituting: $x_\mathrm{l} - (1,0)$ by $x_\mathrm{r} + (1,0)$, "left" by "right" and "right" by "left".

Proof If $x_\mathrm{l} \in \partial_\mathrm{l}\Lambda$ then by (2.7) and (A.2) the vector describing the boundary condition on the left side of the line L is: $B_\mathrm{l} = \begin{pmatrix}0 & 1 & \sqrt{ab}\end{pmatrix}$ if a left-dimer is fixed on $x_\mathrm{l}-(1,0)$; $B_\mathrm{l} = (1\ 0\ 0)$ if a right-dimer is fixed on $x_\mathrm{l}-(1,0)$; $B_\mathrm{l} = \begin{pmatrix}1 & 1 & \sqrt{ab}\end{pmatrix}$ if on $x_\mathrm{l}-(1,0)$ there is a h-dimer with free position. Then the estimates for $B_\mathrm{l}\, E_\mathrm{r}^{(i)}$, $i = 1, 2, 3$, are computed using the proposition A.4. □

Lemma A.8 *If $x_\mathrm{l} \in \partial_\mathrm{l}\Lambda$, then as $\beta \to \infty$*

$$B_\mathrm{l}\, E_\mathrm{r}^{(1)} = \begin{cases} \frac{1}{\sqrt{2}}\left(1 - \frac{a}{2}\,(1+o(1))\right) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ has fixed position} \\ \sqrt{2}\left(1 - \frac{a}{2}\,(1+o(1))\right) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ has free position} \end{cases}$$

$$B_\mathrm{l}\, E_\mathrm{r}^{(2)} = \begin{cases} -\frac{1}{\sqrt{2}}\left(1 + \frac{a}{2}\,(1+o(1))\right) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ is fixed to the left} \\ \frac{1}{\sqrt{2}}\left(1 + \frac{a}{2}\,(1+o(1))\right) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ is fixed to the right} \\ \frac{ab}{2\sqrt{2}}\,(1+o(1)) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ has free position} \end{cases}$$

$$B_\mathrm{l}\, E_\mathrm{r}^{(3)} = \begin{cases} -a^2\sqrt{ab}\,(1+o(1)) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ is fixed to the left} \\ -a\sqrt{ab}\,(1+o(1)) & \text{if the h-dimer on } x_\mathrm{l}-(1,0) \text{ is fixed to the right or free} \end{cases}$$

If $x_\mathrm{r} \in \partial_\mathrm{r}\Lambda$, then the same estimates hold respectively for $E_\mathrm{l}^{(1)}B_\mathrm{r}$, $E_\mathrm{l}^{(2)}B_\mathrm{r}$, $E_\mathrm{l}^{(3)}B_\mathrm{r}$ after substituting: $x_\mathrm{l}-(1,0)$ by $x_\mathrm{r}+(1,0)$, "left" by "right" and "right" by "left".

Proof If $x_\mathrm{l} \in \partial_\mathrm{l}\Lambda$ then by (2.7) and (A.2) the vector describing the boundary condition on the left side of the line L is: $B_\mathrm{l} = \begin{pmatrix}0 & 1 & \sqrt{ab}\end{pmatrix}$ if a left-dimer is fixed on $x_\mathrm{l}-(1,0)$; $B_\mathrm{l} = (1\ 0\ 0)$ if a right-dimer is fixed on $x_\mathrm{l}-(1,0)$; $B_\mathrm{l} = \begin{pmatrix}1 & 1 & \sqrt{ab}\end{pmatrix}$ if on $x_\mathrm{l}-(1,0)$ there is a h-dimer with free position. Then the estimates for $B_\mathrm{l}\, E_\mathrm{r}^{(i)}$, $i = 1, 2, 3$, are computed using the proposition A.4. □

Proposition A.9 *Denote by $o(1)$ any function $\omega(\beta, \mu_\mathrm{h}, J)$ that goes to zero as $\beta \to \infty$ and does not depend on the choice of the line L nor on Λ. Then for every line $L \in \mathscr{L}_\Lambda(\cup_j S_j)$, $S_j \in \mathscr{S}_\Lambda$ pairwise disconnected, $\Lambda \subset \mathbb{Z}^2$ finite, it holds*

$$|R_L| \,\le\, e^{-m|L|}\,\gamma_L \tag{A.8}$$

where the quantity γ_L can be chosen as follows:

$$\gamma_L := \begin{cases} \left(\frac{e^{-\beta J}}{2} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}|L|}\right)(1+o(1)) & \text{if } x_\mathrm{l} \in \cup_i\partial_\mathrm{r}^\mathrm{ext}S_i\,,\ x_\mathrm{r} \in \cup_i\partial_\mathrm{l}^\mathrm{ext}S_i \\ \frac{e^{-\beta\frac{J}{2}}}{\sqrt{2}}\,(1+o(1)) & \text{if } x_\mathrm{l} \in \cup_i\partial_\mathrm{r}^\mathrm{ext}S_i\,,\ x_\mathrm{r} \in \cup_i\partial_\mathrm{r}\Lambda \\ & \text{or vice versa } x_\mathrm{l} \in \cup_i\partial_\mathrm{l}\Lambda\,,\ x_\mathrm{r} \in \cup_i\partial_\mathrm{l}^\mathrm{ext}S_i \\ 1+o(1) & \text{if } x_\mathrm{l} \in \partial_\mathrm{l}\Lambda\,,\ x_\mathrm{r} \in \partial_\mathrm{r}\Lambda \end{cases} \tag{A.9}$$

Proof • Suppose $x_\mathrm{l} \in \partial_\mathrm{r}^\mathrm{ext}S_i$ and $x_\mathrm{r} \in \partial_\mathrm{l}^\mathrm{ext}S_j$. The definition (2.9) and the corollary A.2 give

$$\begin{aligned} \lambda_1\, R_L \;&=\; \frac{Z_L}{\lambda_1^{|L|-1}} \;-\; B_\mathrm{l}E_\mathrm{r}^{(1)}E_\mathrm{l}^{(1)}B_\mathrm{r} \\ &=\; \left(\frac{\lambda_2}{\lambda_1}\right)^{|L|-1} B_\mathrm{l}E_\mathrm{r}^{(2)}E_\mathrm{l}^{(2)}B_\mathrm{r} \;+\; \left(\frac{\lambda_3}{\lambda_1}\right)^{|L|-1} B_\mathrm{l}E_\mathrm{r}^{(3)}E_\mathrm{l}^{(3)}B_\mathrm{r}\,. \end{aligned}$$

By the lemma A.5 $|\lambda_3/\lambda_1| \le a\,|\lambda_2/\lambda_1|$ when β is sufficiently large. Therefore, using also the estimates of lemma A.6, one finds

$$|R_L| \;\le\; \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} \left(\frac{b}{2}\,(1+o(1)) \;+\; a^{|L|-1}\,a\,(1+o(1))\right) \\ = \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} \left(\frac{b}{2} + a^{|L|}\right)(1+o(1))\;.$$

• Suppose now $x_\mathrm{l} \in \partial_\mathrm{r}^{\mathrm{ext}} S_j$ and $x_\mathrm{r} \in \partial_\mathrm{r}\Lambda$. The definition (2.9) and the corollary A.2 give

$$\lambda_1^{1/2}\,R_L \;=\; \frac{Z_L}{\lambda_1^{|L|-1} E_\mathrm{l}^{(1)} B_\mathrm{r}} \;-\; B_\mathrm{l} E_\mathrm{r}^{(1)} \\ = \left(\frac{\lambda_2}{\lambda_1}\right)^{|L|-1} \frac{B_\mathrm{l} E_\mathrm{r}^{(2)} E_\mathrm{l}^{(2)} B_\mathrm{r}}{E_\mathrm{l}^{(1)} B_\mathrm{r}} \;+\; \left(\frac{\lambda_3}{\lambda_1}\right)^{|L|-1} \frac{B_\mathrm{l} E_\mathrm{r}^{(3)} E_\mathrm{l}^{(3)} B_\mathrm{r}}{E_\mathrm{l}^{(1)} B_\mathrm{r}}\;.$$

By the lemma A.5 $|\lambda_3/\lambda_1| \le a\,|\lambda_2/\lambda_1|$ when β is sufficiently large. Therefore, using also the estimates of lemmas A.6, A.8, one obtains

$$|R_L| \;\le\; \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} \gamma$$

with

$$\gamma \;=\; \begin{cases} \text{if fixed h-dimer on } x_\mathrm{r}+(1,0) & \frac{\sqrt{b}}{\sqrt{2}}\,(1+o(1)) \;+\; a^{|L|-1}\,O(a^2\sqrt{b}\,) \\ \text{if free h-dimer on } x_\mathrm{r}+(1,0) & \frac{ab\sqrt{b}}{4\sqrt{2}}\,(1+o(1)) \;+\; a^{|L|-1}\,\frac{a^2 b}{\sqrt{2}}\,(1+o(1)) \end{cases}$$

$$= \begin{cases} \frac{\sqrt{b}}{\sqrt{2}}\,(1+o(1)) \\ \left(\frac{ab\sqrt{b}}{4\sqrt{2}} + \frac{a^{|L|+1} b}{\sqrt{2}}\right)(1+o(1)) \end{cases} \;\le\; \frac{\sqrt{b}}{\sqrt{2}}\,(1+o(1))\;.$$

• Suppose now $x_\mathrm{l} \in \partial_\mathrm{l}\Lambda$ and $x_\mathrm{r} \in \partial_\mathrm{r}\Lambda$. The definition (2.9) and the corollary A.2 give

$$R_L \;=\; \frac{Z_L}{\lambda_1^{|L|-1} B_\mathrm{l} E_\mathrm{r}^{(1)} E_\mathrm{l}^{(1)} B_\mathrm{r}} \;-\; 1 \\ = \left(\frac{\lambda_2}{\lambda_1}\right)^{|L|-1} \frac{B_\mathrm{l} E_\mathrm{r}^{(2)} E_\mathrm{l}^{(2)} B_\mathrm{r}}{B_\mathrm{l} E_\mathrm{r}^{(1)} E_\mathrm{l}^{(1)} B_\mathrm{r}} \;+\; \left(\frac{\lambda_3}{\lambda_1}\right)^{|L|-1} \frac{B_\mathrm{l} E_\mathrm{r}^{(3)} E_\mathrm{l}^{(3)} B_\mathrm{r}}{B_\mathrm{l} E_\mathrm{r}^{(1)} E_\mathrm{l}^{(1)} B_\mathrm{r}}\;.$$

By the lemma A.5 $|\lambda_3/\lambda_1| \le a\,|\lambda_2/\lambda_1|$ when β is sufficiently large. Therefore, using also the estimates of lemma A.8, one obtains

$$|R_L| \;\le\; \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} \gamma$$

with

$$\gamma \;=\; \begin{cases} \text{if fixed h-b.c. on both sides} & 1 + 2a\,(1+o(1)) \;+\; a^{|L|-1}\,O(a^3 b) \\ \text{if fixed h-b.c. on one side,} & \frac{ab}{4}\,(1+o(1)) \;+\; a^{|L|-1}\,O(a^3 b) \\ \text{free h-b.c. on the other one} & \\ \text{if free h-b.c. on both sides} & \frac{a^2 b^2}{8}\,(1+o(1)) \;+\; a^{|L|-1}\,\frac{a^3 b}{2}\,(1+o(1)) \end{cases}$$

$$= \begin{cases} 1 + 2a\,(1+o(1)) \\ \frac{ab}{4}\,(1+o(1)) \\ \left(\frac{a^2 b^2}{8} + \frac{a^{|L|+2} b}{2}\right)(1+o(1)) \end{cases} \;\le\; 1 + o(1)\;.$$

□

B Appendix: Cluster Expansion

In this Appendix we state the main results about the general theory of cluster expansion used in this paper. The condition that we adopt to guarantee the convergence of the expansion is due to Kotecky-Preiss [11]. For a modern proof we refer to [17].

Let $\mathscr{P}$ be a finite set, called the *set of polymers*. Let $\varrho : \mathscr{P} \to \mathbb{C}$, called the *polymer activity*, and $\delta : \mathscr{P} \times \mathscr{P} \to \{0, 1\}$, called the *polymer hard-core interaction*, such that $\delta(P, P) = 0$ and $\delta(P, P') = \delta(P', P)$ for all $P, P' \in \mathscr{P}$. Consider the *polymer partition function*:

$$\begin{aligned} \mathscr{Z} &:= \sum_{\mathscr{P}' \subseteq \mathscr{P}} \prod_{P \in \mathscr{P}'} \varrho(P) \prod_{\substack{P, P' \in \mathscr{P}' \\ P \neq P'}} \delta(P, P') \\ &= \sum_{q \geq 0} \frac{1}{q!} \sum_{P_1, \ldots, P_q \in \mathscr{P}} \prod_{t=1}^{q} \varrho(P_t) \prod_{t<s} \delta(P_t, P_s) \ . \end{aligned} \tag{B.1}$$

A family of polymers $(P_1, \ldots, P_q)$ is called *compatible* if $\delta(P_t, P_s) = 1$ for all $t \neq s$; otherwise it is called *incompatible*. Observe that in the partition function $\mathscr{Z}$ only the compatible families of polymers give non-zero contributions.
A family of polymers $(P_1, \ldots, P_q)$ is called a *cluster* if the graph with vertex set $\{1, \ldots, q\}$ and edge set $\{(t, s) \,|\, \delta(P_t, P_s) = 0\}$ is connected.

Theorem B.1 *Suppose that there exists* $a \colon \mathscr{P} \to [0, \infty[$ *, called size function, such that the Kotecky-Preiss condition is satisfied, namely:*

$$\sum_{\substack{P \in \mathscr{P} \\ \delta(P, P^*) = 0}} |\varrho(P)| \, e^{a(P)} \,\leq\, a(P^*) \quad \forall\, P^* \in \mathscr{P} \ . \tag{B.2}$$

Then:

$$\log \mathscr{Z} \,=\, \sum_{q \geq 0} \frac{1}{q!} \sum_{P_1, \ldots, P_q \in \mathscr{P}} \left(\prod_{t=1}^{q} \varrho(P_t) \right) u(P_1, \ldots, P_q) \tag{B.3}$$

where the series on the r.h.s. is absolutely convergent and

$$u(P_1, \ldots, P_q) \,:=\, \sum_{\substack{G=(V,E) \text{ connected graph} \\ V = \{1, \ldots, q\} \\ E \subseteq \{(t,s) \,|\, \delta(P_t, P_s) = 0\}}} (-1)^{|E|} \ . \tag{B.4}$$

Moreover, for all $\mathscr{E} \subseteq \mathscr{P}$

$$\sum_{q \geq 0} \frac{1}{q!} \sum_{\substack{P_1, \ldots, P_q \in \mathscr{P} \\ \exists\, t \colon P_t \in \mathscr{E}}} \left| \prod_{t=1}^{q} \varrho(P_t) \right| |u(P_1, \ldots, P_q)| \,\leq\, \sum_{\substack{P \in \mathscr{P} \\ P \in \mathscr{E}}} |\varrho(P)| \, e^{a(P)} \ . \tag{B.5}$$

It is worth to observe that if $(P_1, \ldots, P_q)$ is not a cluster then $u(P_1, \ldots, P_q) = 0$. Therefore only the clusters of polymers (that are infinitely many) give non-zero contributions to the expansion (B.3) of $\log \mathscr{Z}$.

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References

1. A. Bovier, M. Zahradnik, “A simple inductive approach to the problem of convergence of cluster expansions of polymer models”, *J. Stat. Phys.* **100**(3-4), 765-778, 2000
2. P. Contucci, “An analyticity bound for two-dimensional Ising model at low temperature”, *J. Stat. Phys.* **82**(5-6), 1647-1657, 1993
3. M. Disertori, A. Giuliani, “The Nematic Phase of a System of Long Hard Rods”, *Commun. Math. Phys.* **323**(1), 143-175, 2013
4. R.L. Dobrushin, “Estimates of Semi-invariants for the Ising Model at Low Temperatures”, in *Topics in Statistical and Theoretical Physics. F.A. Berezin Memorial Volume*, pp. 59-81, eds. R.L. Dobrushin, R.A. Minlos, M.A. Shubin, A.M. Vershik, American Mathematical Society, 1995
5. R. Fernández, A. Procacci, “Cluster Expansion for Abstract Polymer Models. New Bounds from an Old Approach”, *Commun. Math. Phys.* **274**(1), 123-140, 2007
6. G. Gallavotti, A. Martin-Löf, S. Miracle-Solé, “Some problems connected with the description of Coexisting Phases at Low Temperatures in the Ising Model”, *Lecture Notes in Physics* **20**, 162-204, 1973
7. C. Gruber, H. Kunz, “General Properties of Polymer Systems”, *Commun. Math. Phys.* **22**, 133-161, 1971
8. O.J. Heilmann, E.H. Lieb, “Monomers and Dimers”, *Phys. Rev. Lett.* **24**(25), 1412-1414, 1970
9. O.J. Heilmann, E.H. Lieb, “Theory of Monomer-Dimer Systems”, *Commun. Math. Phys.* **25**(3), 190-232, 1972
10. O.J. Heilmann, E.H. Lieb, “Lattice Models for Liquid Crystals”, *J. Stat. Phys.* **20**(6), 679-693, 1979
11. R. Kotecky, D. Preiss, “Cluster Expansion for Abstract Polymer Models”, *Commun. Math. Phys.* **103**(3), 491-498, 1986
12. I. Letawe, *Le Modèle de Cristaux Liquides de Heilmann et Lieb*, theses in Physics at Université Catholique de Louvain, supervisor J. Bricmont, 1994
13. S. Miracle-Solé, “On the Convergence of Cluster Expansions”, *Phys. A* **279**, 244-249, 2000
14. L. Onsager, “The effects of shape on the interaction of colloidal particles”, *Ann. New York Academy of Sciences* **51**, 627-659, 1949
15. S. Papanikolaou, D. Charrier, E. Fradkin, “Ising Nematic Fluid Phase of Hard-Core Dimers on the Square Lattice”, preprint *http://arxiv.org/abs/1310.4173*, 2014
16. D. Ruelle, *Statistical Mechanics: Rigorous Results*, chap. 4, pp. 71-107, W.A. Benjamin Inc., 1969
17. D. Ueltschi, “Cluster Expansion & Correlation Functions”, *Moscow Math. J.* **4**, 511522, 2004